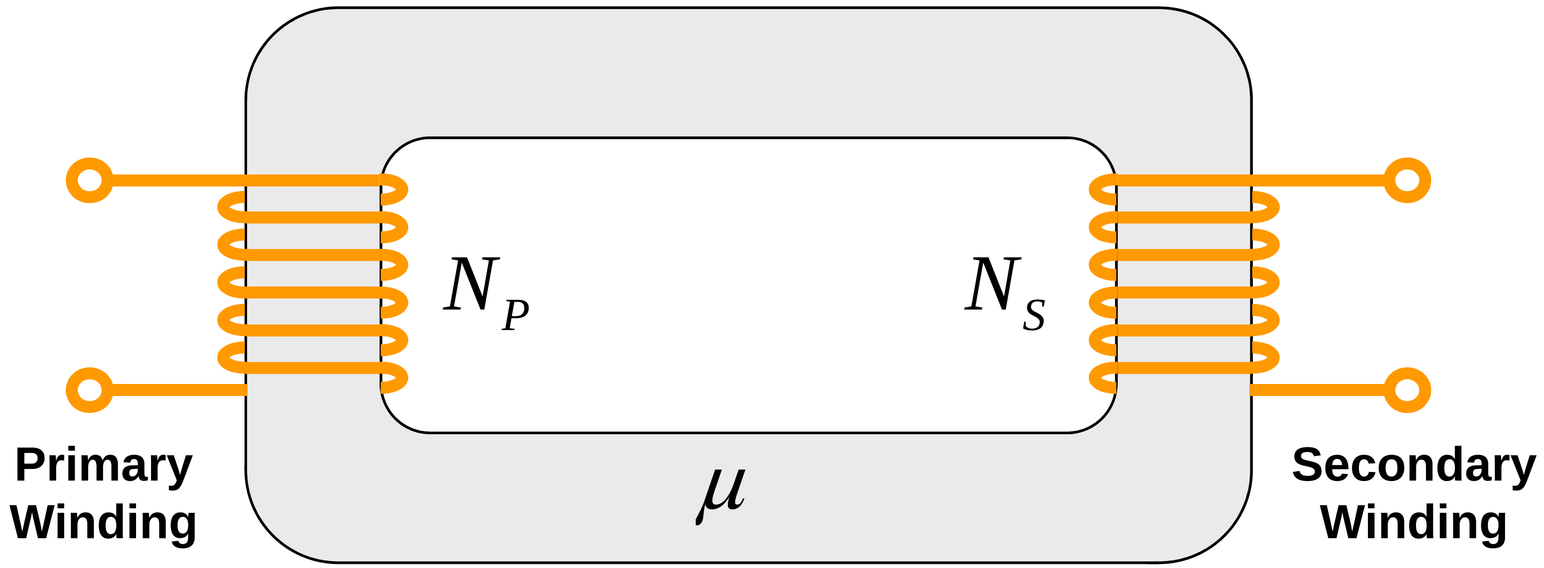
Transformers

Revised October 6, 2008

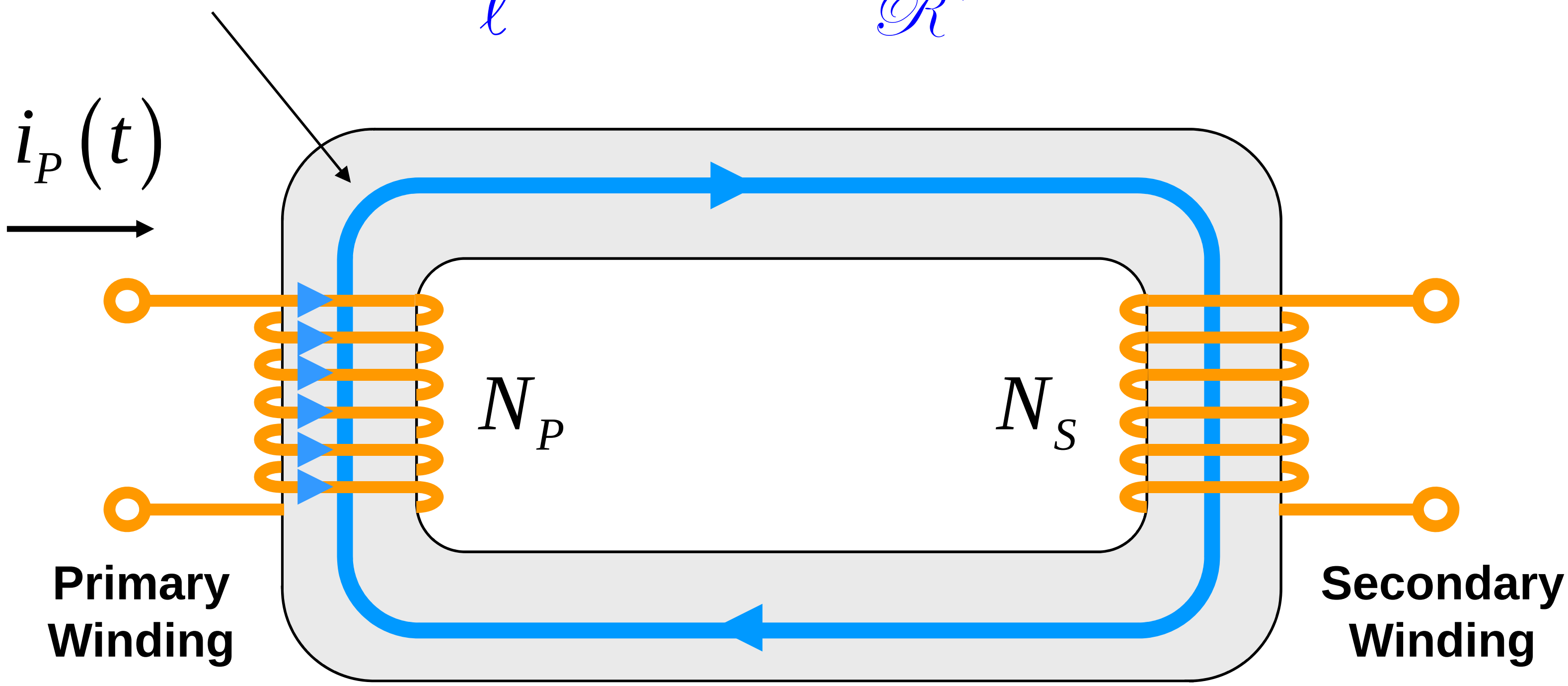
The Ideal Transformer



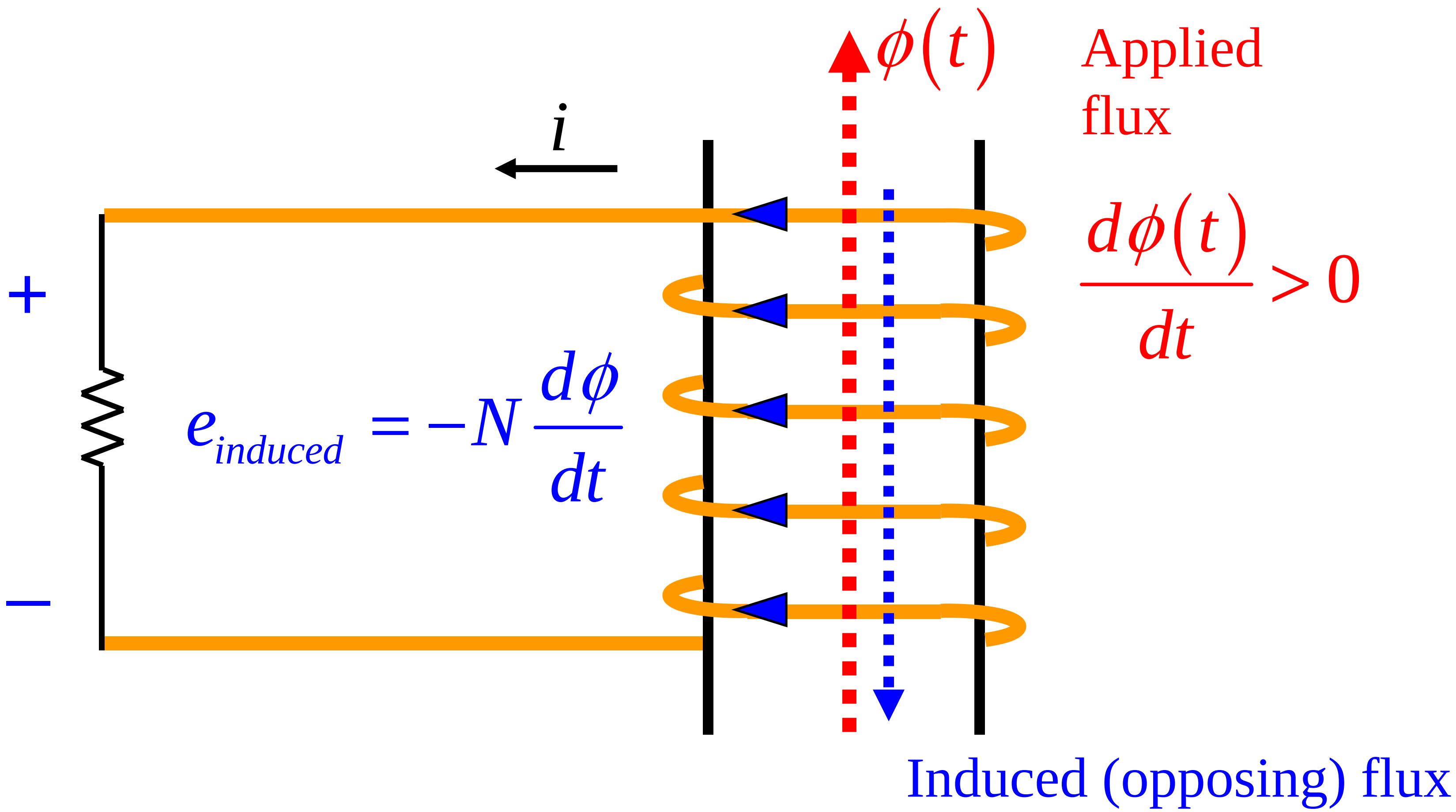
Basic Transformer

The Ideal Transformer - A primary current produces a flux in the core.

$$\phi_P = N_P \frac{\mu A}{\ell} i_P(t) = \frac{N_P i_P(t)}{\mathscr{R}}$$

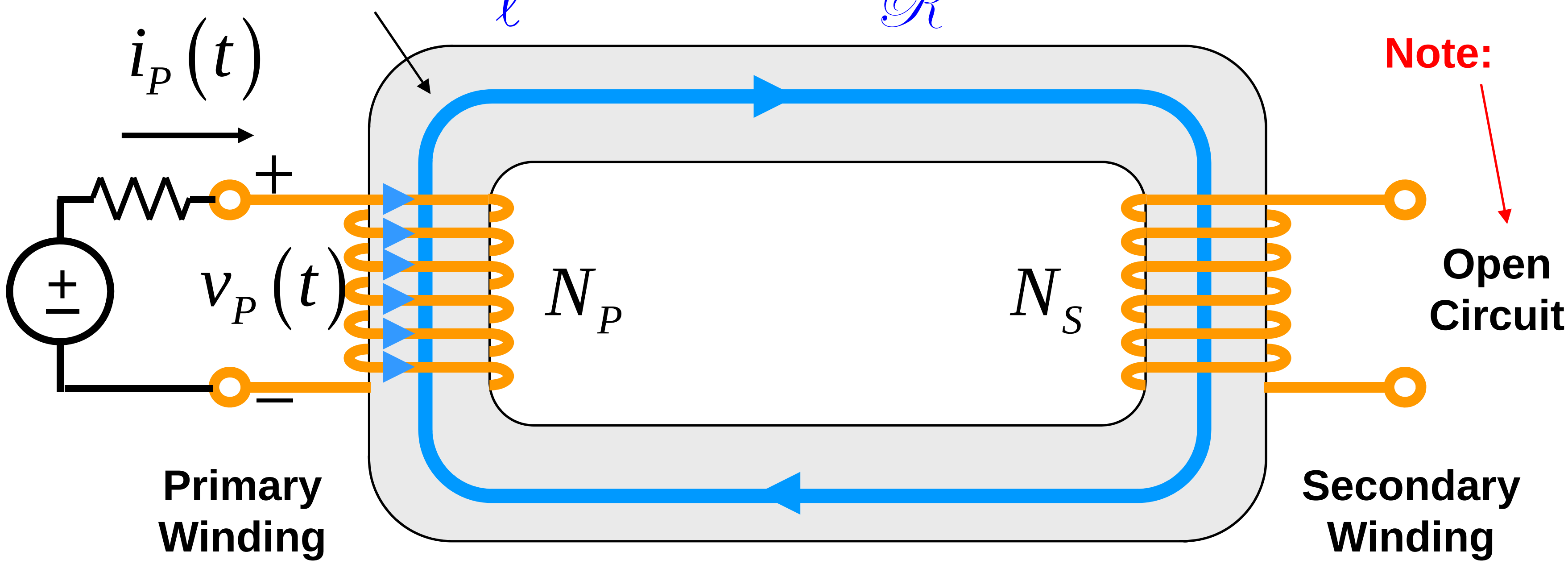


Recall Slide Set 1:



Ideal Transformer – The primary current gives rise to an *induced voltage across the primary winding*.

$$\phi_P = N_P \frac{\mu A}{\ell} i_P(t) = N_P \frac{1}{\mathscr{R}} i_P(t)$$



$$e_{induced_P}(t) = v_P(t) = N_P \frac{d\phi_P}{dt} = \frac{N_P^2}{\mathscr{R}} \frac{di_P(t)}{dt}$$

Self-Inductance of Primary – the output (secondary) is open-circuited

$$v_P(t) = N_P \frac{d\phi_P}{dt} = \frac{N_P^2}{\mathscr{R}} \frac{di_P(t)}{dt}$$

$$= L_P \frac{di_P(t)}{dt}, \quad L_P = \frac{N_P^2}{\mathscr{R}} = \mathscr{P}\ N_P^2$$

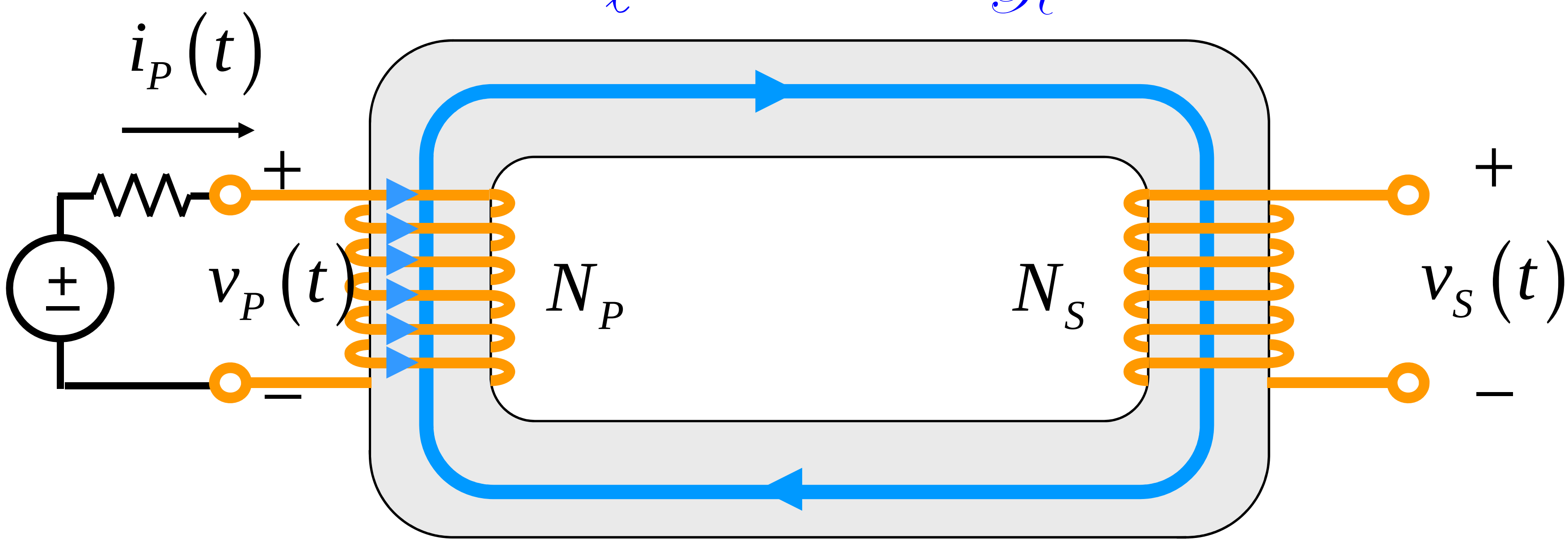
similarly,

$$v_S(t) = N_S \frac{d\phi_S}{dt} = \frac{N_S^2}{\mathscr{R}} \frac{di_S(t)}{dt}$$

$$= L_S \frac{di_S(t)}{dt}, \quad L_S = \frac{N_S^2}{\mathscr{R}} = \mathscr{P}\ N_S^2$$

Ideal Transformer – the flux produced by the primary winding is coupled to the secondary winding *and induces a voltage across the secondary.*

$$\phi_P = N_P \frac{\mu A}{\ell} i_P(t) = N_P \frac{1}{\mathscr{R}} i_P(t)$$



$$e_{induced_S}(t) = v_S(t) = N_S \frac{d\phi_P}{dt} = \frac{N_p N_S}{\mathscr{R}} \frac{di_P(t)}{dt}$$

Mutual Inductance

$$v_S(t) = N_S \frac{d\phi_P}{dt} = \frac{N_p N_S}{\mathscr{R}} \frac{di_P(t)}{dt}$$

$$= M \frac{di_P(t)}{dt}, \quad M = \frac{N_p N_S}{\mathscr{R}}$$

$$L_P = \frac{N_P^2}{\mathscr{R}}, \quad L_S = \frac{N_S^2}{\mathscr{R}} \Rightarrow M = \sqrt{L_P L_S}$$

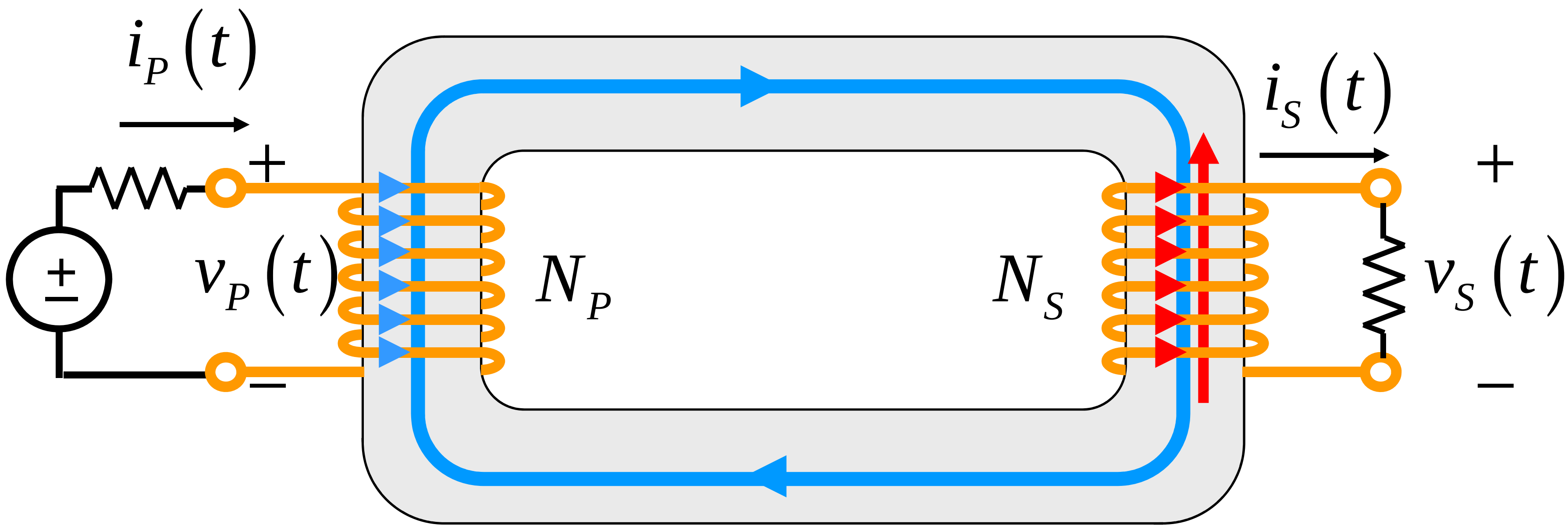
Ideal Transformer

$$\frac{v_P(t)}{v_S(t)} = \frac{\dfrac{N_P^2}{\mathscr{R}}\dfrac{di_P(t)}{dt}}{\dfrac{N_p N_S}{\mathscr{R}}\dfrac{di_P(t)}{dt}} = \frac{N_p}{N_S} = \sqrt{\frac{L_P}{L_S}}$$

$$\frac{v_P(t)}{v_S(t)} = \frac{N_p}{N_S} = Turns\ Ratio$$

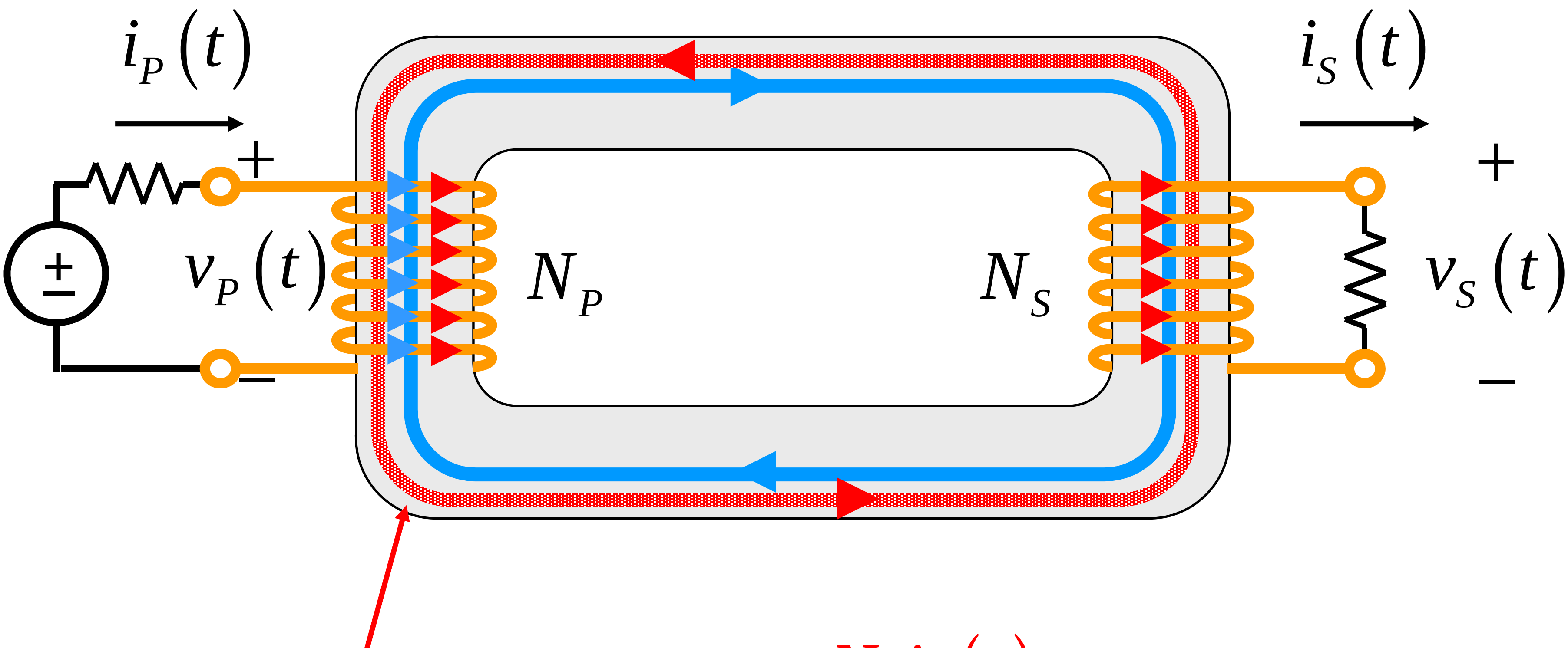
Ideal Transformer – A load across the secondary winding will cause a *current to flow that opposes the change in flux of the primary*.

$$\phi_S = N_S \frac{\mu A}{\ell} i_S(t) = \frac{N_S i_S(t)}{\mathscr{R}}$$



$$e_{induced_S}(t) = v_S(t) = N_S \frac{d\phi_P}{dt} = \frac{N_p N_S}{\mathscr{R}} \frac{di_P(t)}{dt}$$

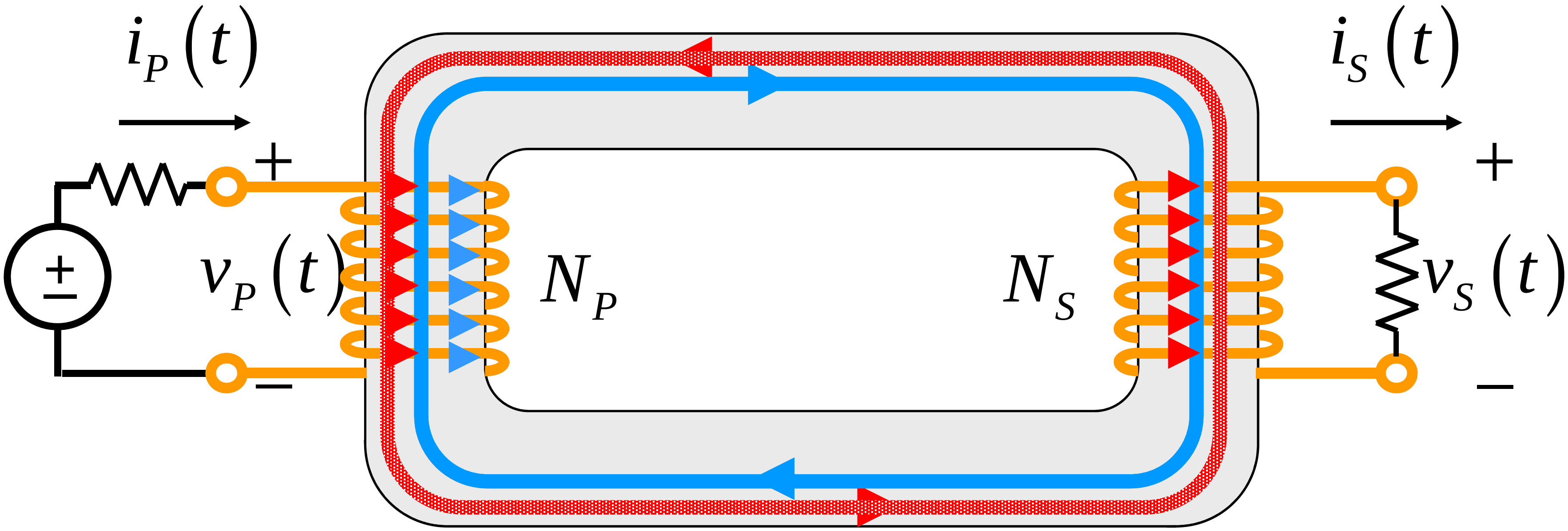
Ideal Transformer – The flux produced by the secondary current will *coupled back to the primary winding*.



$$\phi_S = N_S \frac{\mu A}{\ell} i_S(t) = \frac{N_S i_S(t)}{\mathscr{R}}$$

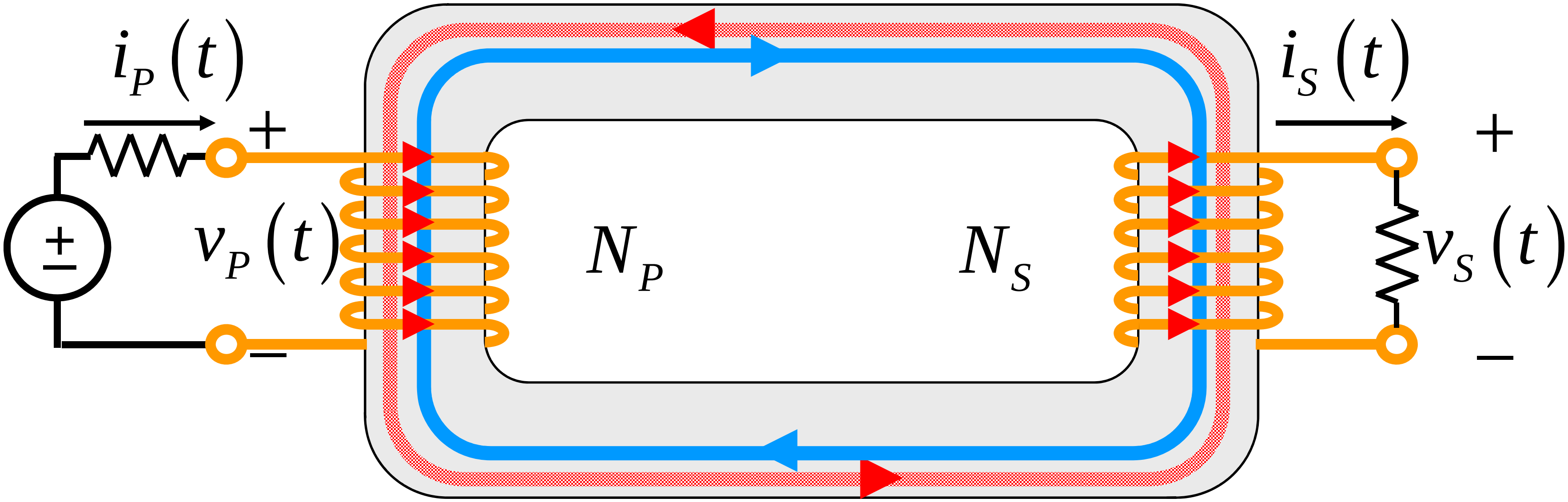
Ideal Transformer – The flux produced by the secondary will be coupled back to the primary winding *inducing a voltage on the primary*.

$$\phi_S = N_S \frac{\mu A}{\ell} i_S(t) = N_S \frac{1}{\mathscr{R}} i_S(t)$$



$$e_{induced_P}(t) = N_P \frac{d\phi_S}{dt} = \frac{N_P N_S}{\mathscr{R}} \frac{di_S(t)}{dt}$$

The relationship between the primary and secondary currents can be determined from the total *mmf*:



$$mmf_P = N_P i_P(t) = \mathscr{R}\phi_P$$

$$mmf_S = -N_S i_S(t) = -\mathscr{R}\phi_S$$

For the *Ideal* Transformer:

$$total\ mmf:$$

$$\Rightarrow N_P i_P(t) - N_S i_S(t) = \mathscr{R}(\phi_P - \phi_S) = 0$$

$$\mu \rightarrow \infty$$

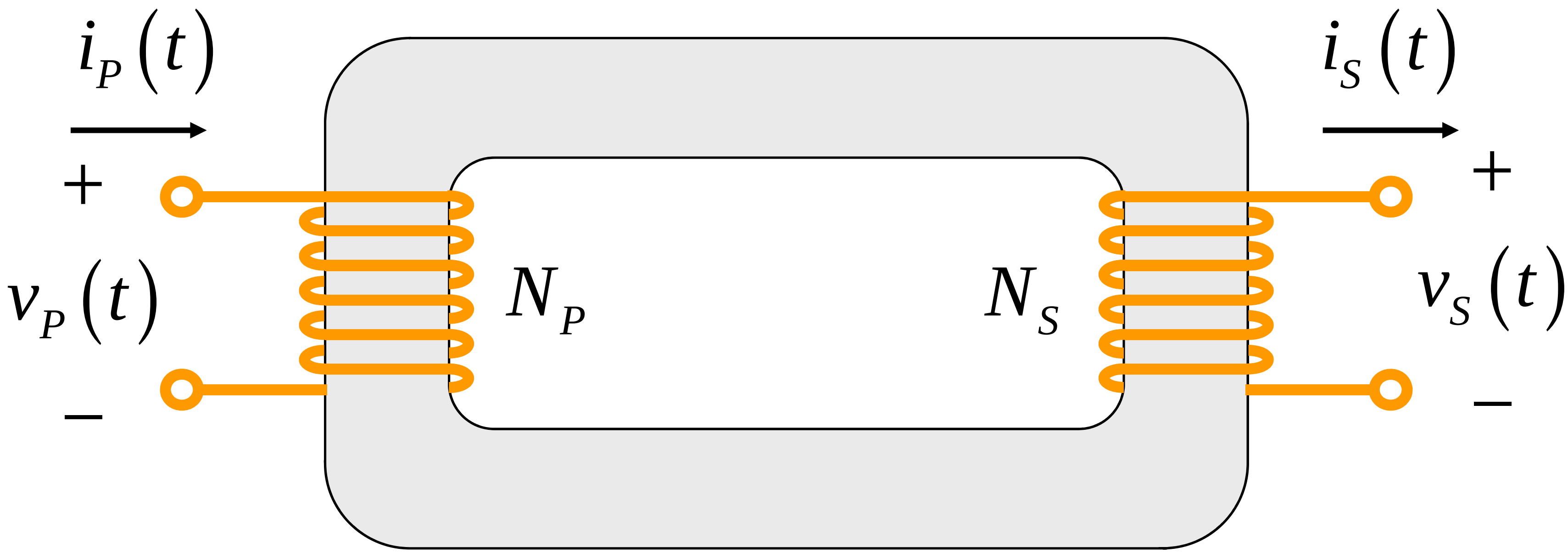
$$\mathscr{R} \rightarrow 0$$

$$\Rightarrow \frac{i_P(t)}{i_S(t)} = \frac{N_S}{N_P},\ \text{Re}\,call: \frac{v_P(t)}{v_S(t)} = \frac{N_P}{N_S}$$

$$total\ \ power:$$

$$\frac{i_P(t)}{i_S(t)}\frac{v_P(t)}{v_S(t)} = \frac{N_S}{N_P}\frac{N_P}{N_S} = 1, \Rightarrow p_P = p_S$$

Ideal Transformer

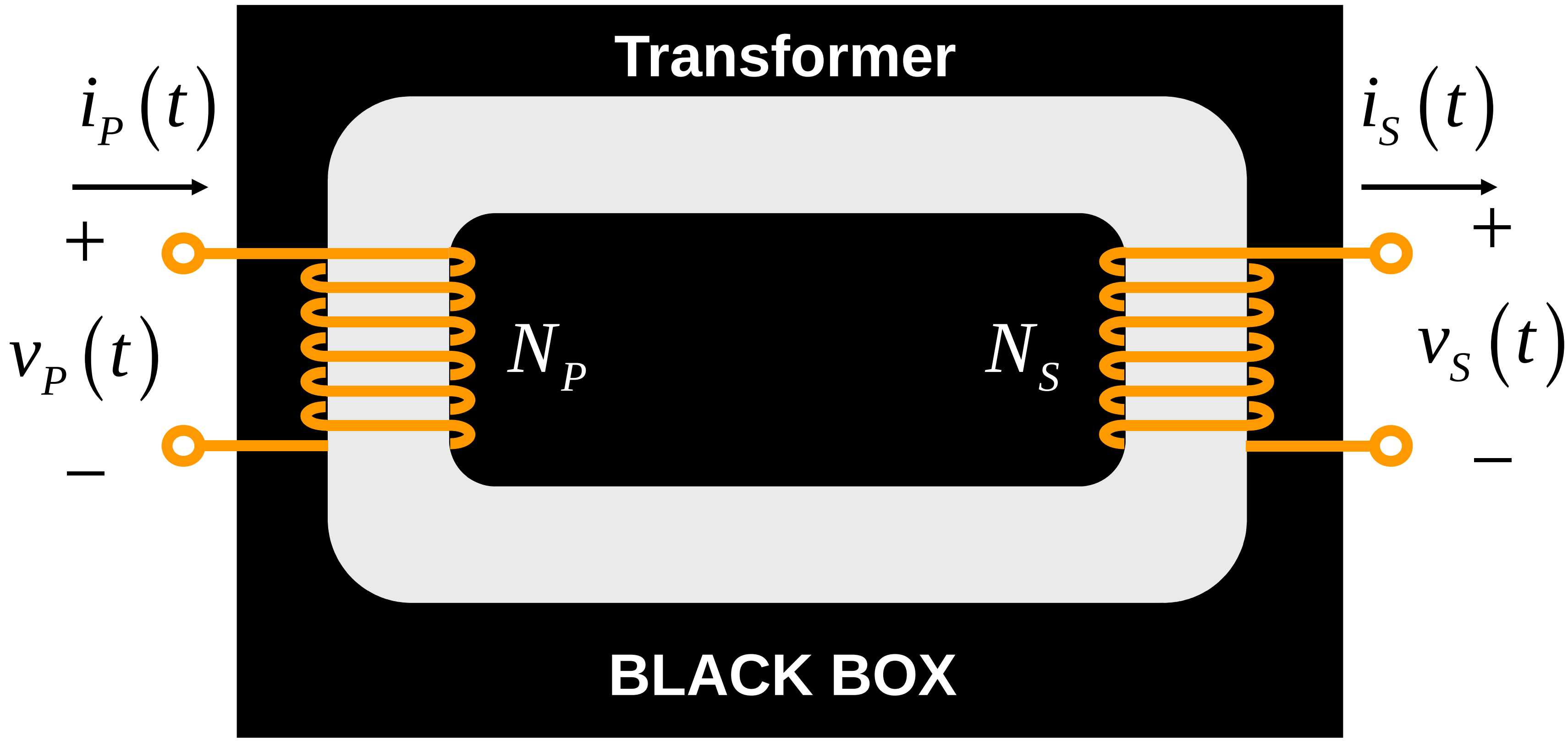


$$v_P(t) = \frac{N_P^2}{\mathscr{R}} \frac{di_P(t)}{dt} + \frac{N_P N_S}{\mathscr{R}} \frac{di_S(t)}{dt}$$

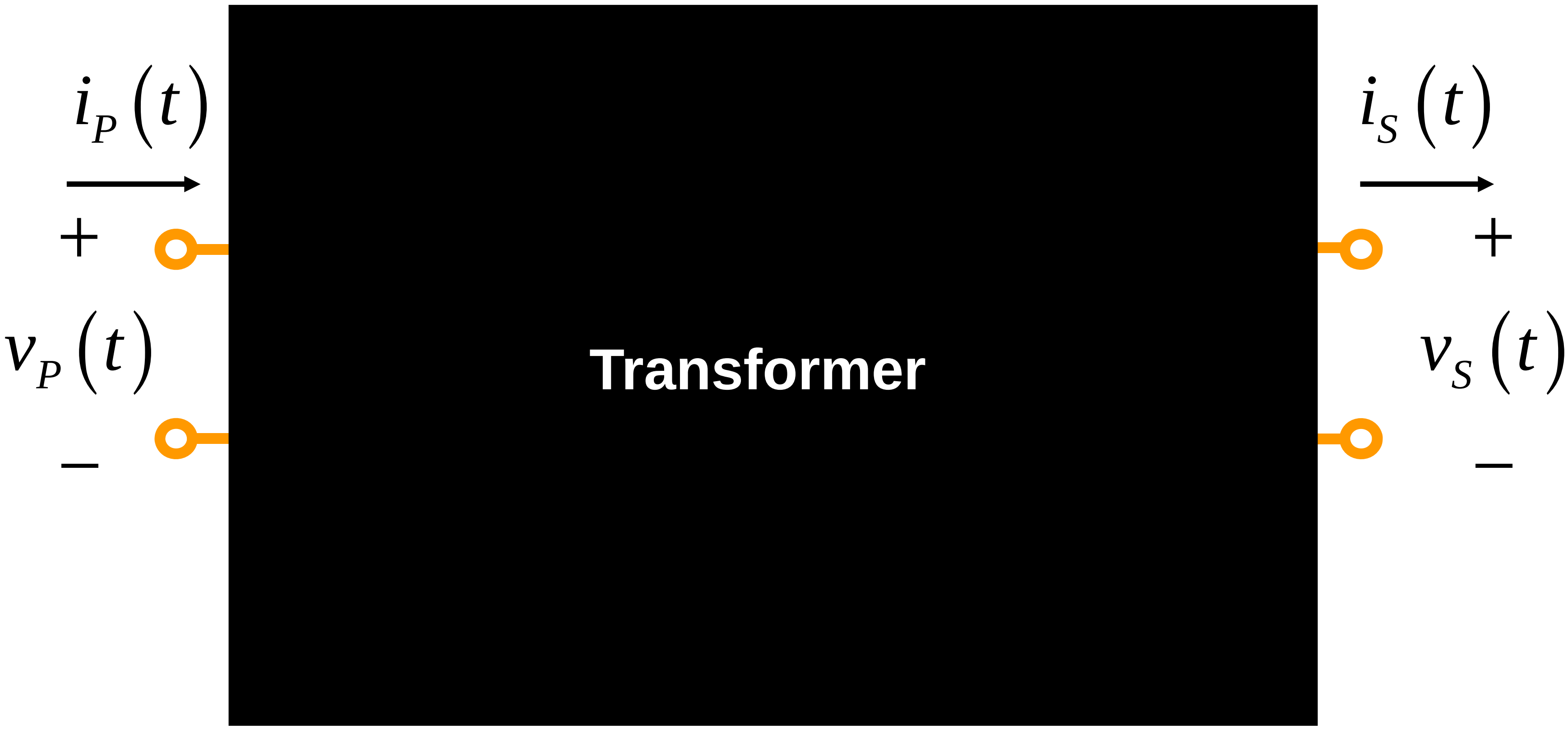
Similarly,

$$v_S(t) = \frac{N_P N_S}{\mathscr{R}} \frac{di_P(t)}{dt} + \frac{N_S^2}{\mathscr{R}} \frac{di_S(t)}{dt}$$

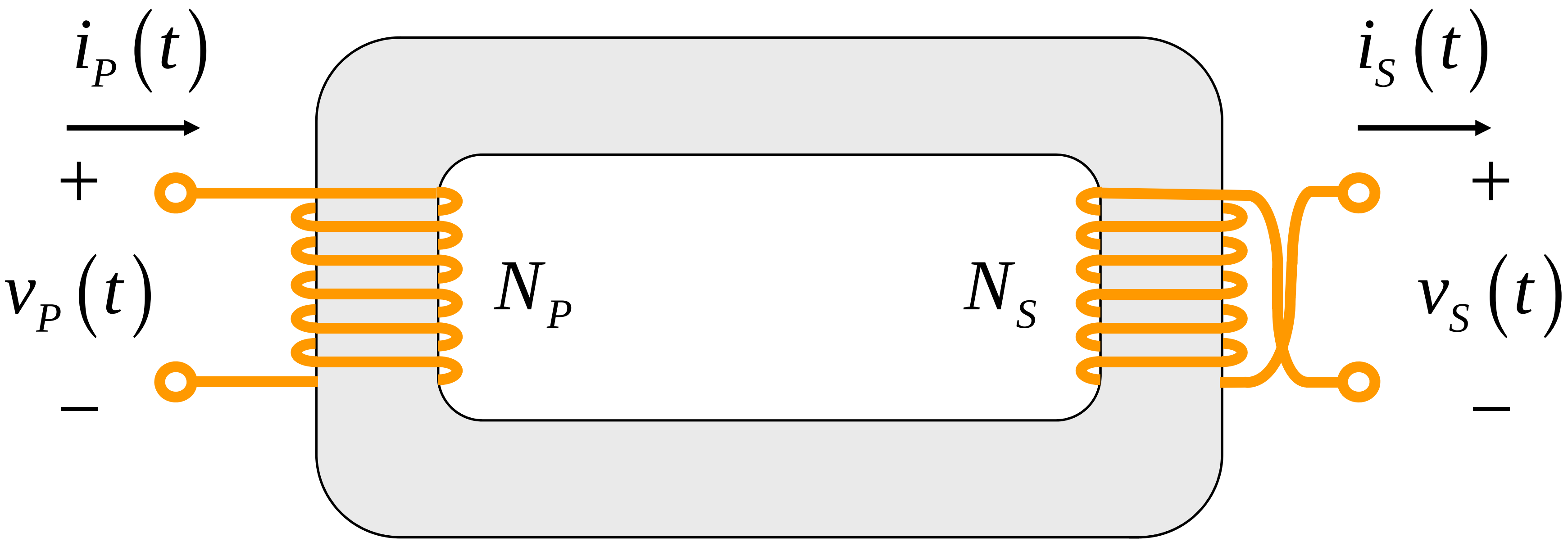
Ideal Transformer



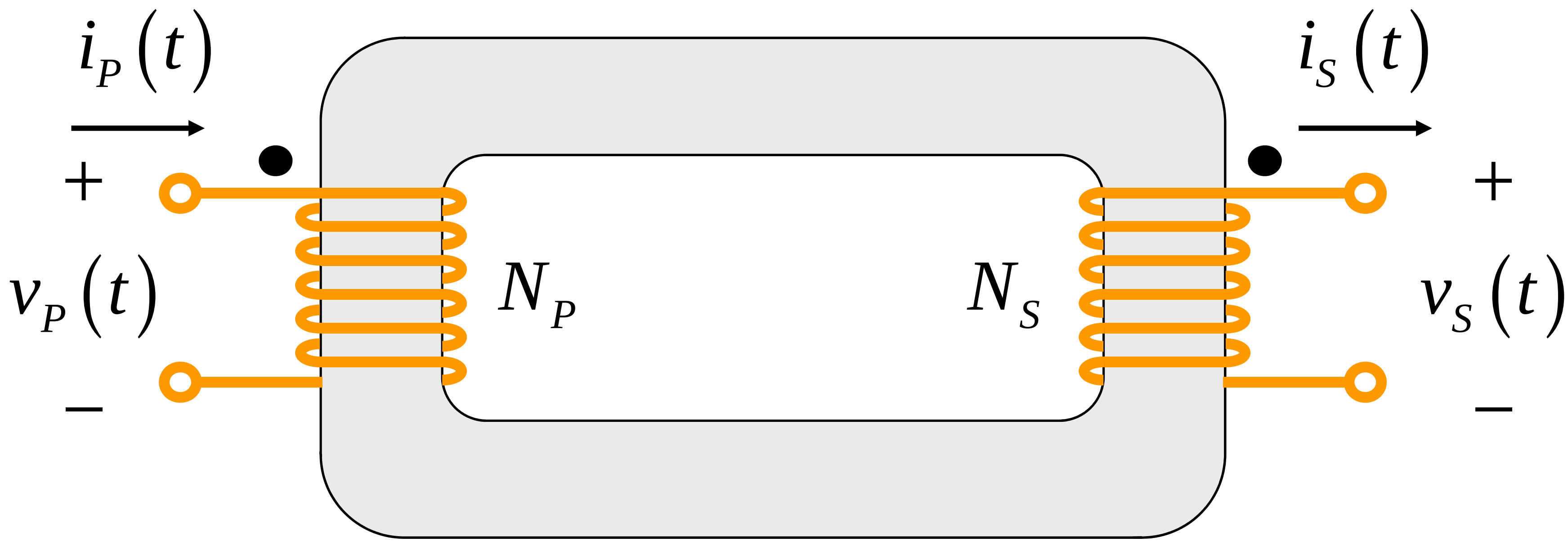
Ideal Transformer



Ideal Transformer

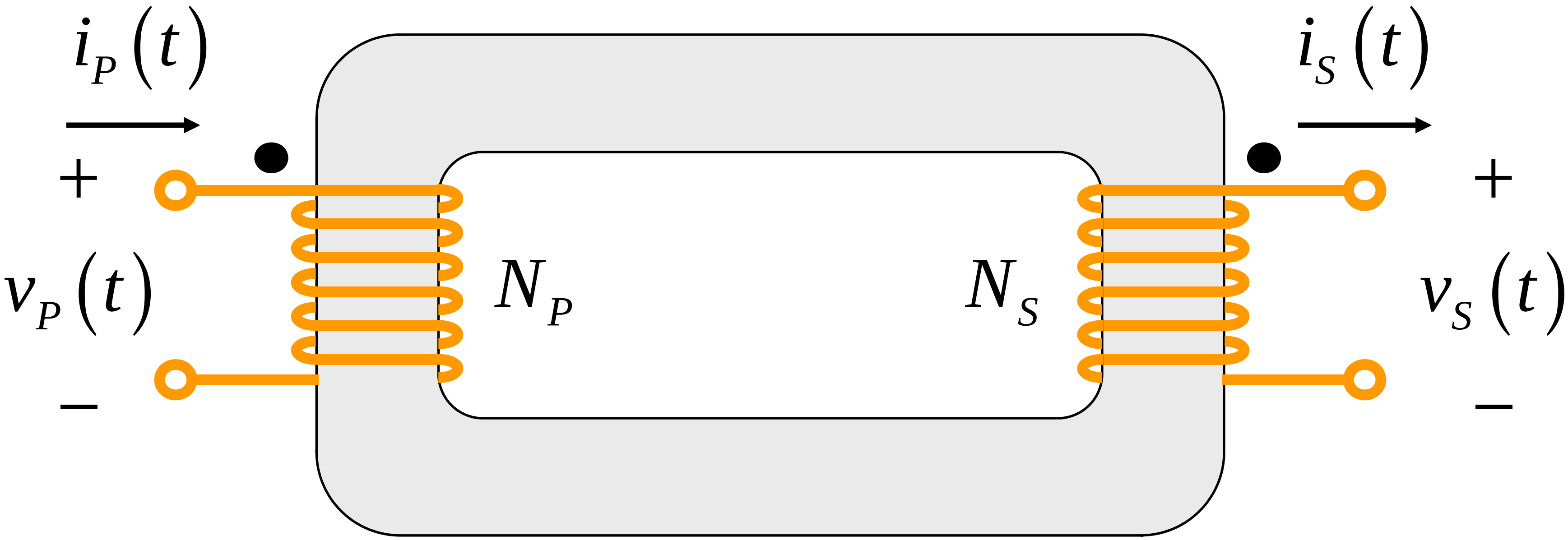


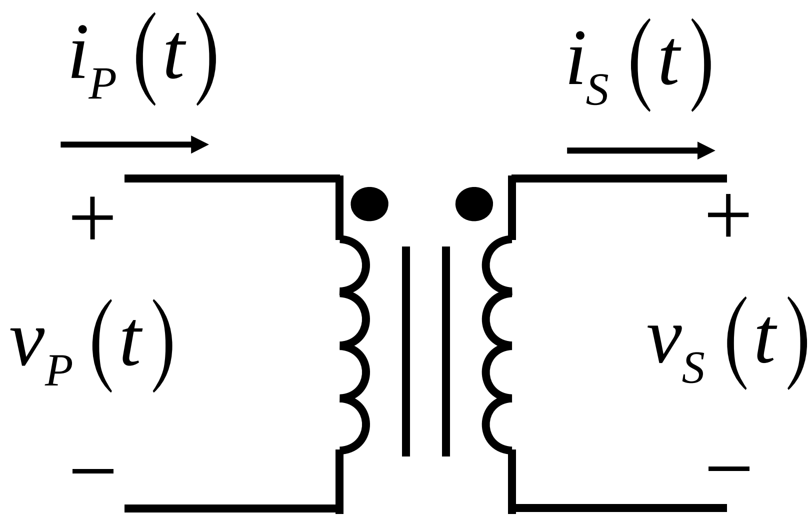
Ideal Transformer



Dot Convention: When the current on the primary/secondary enters the dotted terminal, the polarity of the voltage on the secondary/primary is positive.

Ideal Transformer





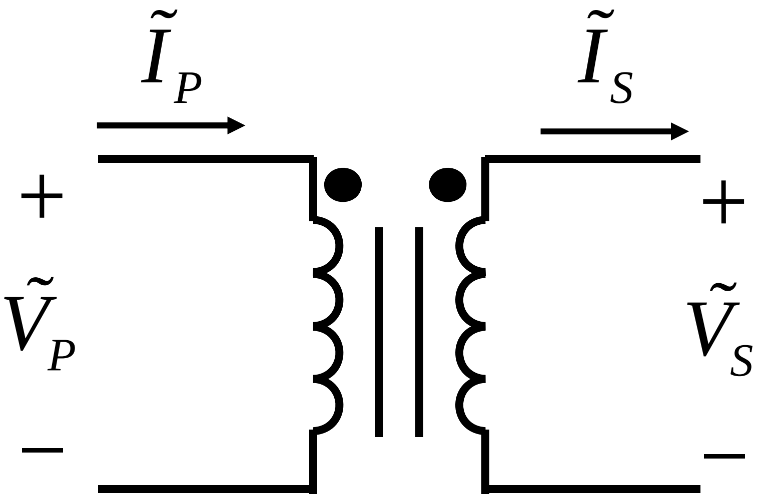
Circuit Representation

Phasor Representation

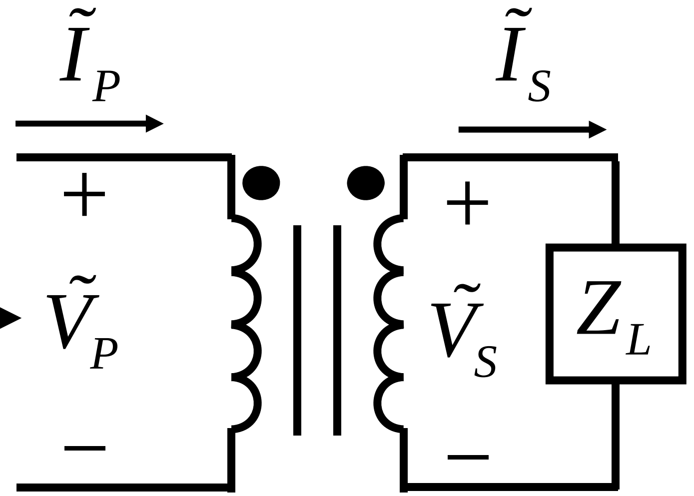
$$\tilde{V}_P = j\omega L_1 \tilde{I}_P + j\omega M \tilde{I}_S$$

$$\tilde{V}_S = j\omega M \tilde{I}_P + j\omega L_2 \tilde{I}_S$$

$$L_1 = \frac{N_P^2}{\mathscr{R}}, \quad L_2 = \frac{N_S^2}{\mathscr{R}}, \quad M = \frac{N_P N_S}{\mathscr{R}}$$



Impedance Transformation



$$Z_{in} = \frac{\tilde{V}_P}{\tilde{I}_P}$$

$$\tilde{V}_S = Z_L \tilde{I}_S$$

$$\frac{\tilde{V}_P}{\tilde{V}_S} = \frac{N_P}{N_S}, \quad \frac{\tilde{I}_P}{\tilde{I}_S} = \frac{N_S}{N_P}, \quad \tilde{V}_S = Z_L \tilde{I}_S$$

$$Z_{in} = \frac{\tilde{V}_P}{\tilde{I}_P} = \frac{\tilde{V}_S \frac{N_P}{N_S}}{\tilde{I}_S \frac{N_S}{N_P}} = \frac{Z_L \tilde{I}_S \frac{N_P}{N_S}}{\tilde{I}_S \frac{N_S}{N_P}} = \left(\frac{N_P}{N_S}\right)^2 Z_L$$

Review EEL 3111 as needed for circuit analysis techniques with transformers. (Chapter 13 in Alexander and Sadiku, Fundamentals of Electric Circuits)

Real (Non-Ideal) Transformers

Revisit Flux Linkage – Note Set 1, Slide 64:

Flux Linkage:

$$e_{induced} = N\frac{d\phi}{dt} = \frac{d\lambda}{dt}$$

Flux Linkage: $\lambda = N\phi$

But each turn does not necessarily link the same amount of flux.

Flux Linkage

Taken from: J.D. Kraus & D.A. Fleisch, Electromagnetics with Applications, Fifth Edition, McGraw-Hill, 1999, page 98.

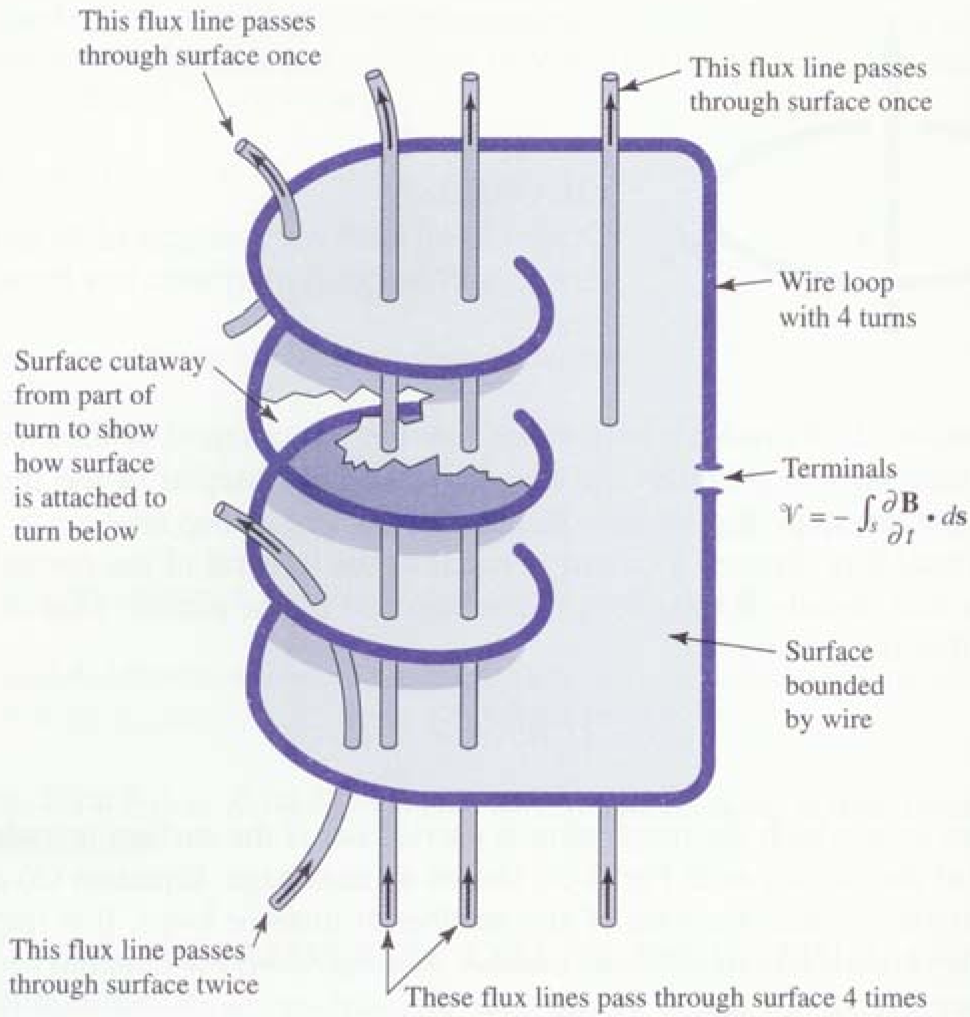


FIGURE 2-50
Circuit with four-turn coil. The wire forms the boundary of a single continuous surface. Part of the surface of one turn has been cut away to show how the surface is bounded by the turn below.

Flux Linkage

$$e_{induced}(t) = N\frac{d\phi}{dt} = \frac{d\lambda}{dt}$$

$$\Rightarrow \lambda = \int e_{induced}(t)\,dt$$

$$\overline{\phi} = \frac{1}{N}\int e_{induced}(t)\,dt$$

↑ Average flux per turn

Hence we really mean: $e_{induced}(t) = N\frac{d\overline{\phi}}{dt}$

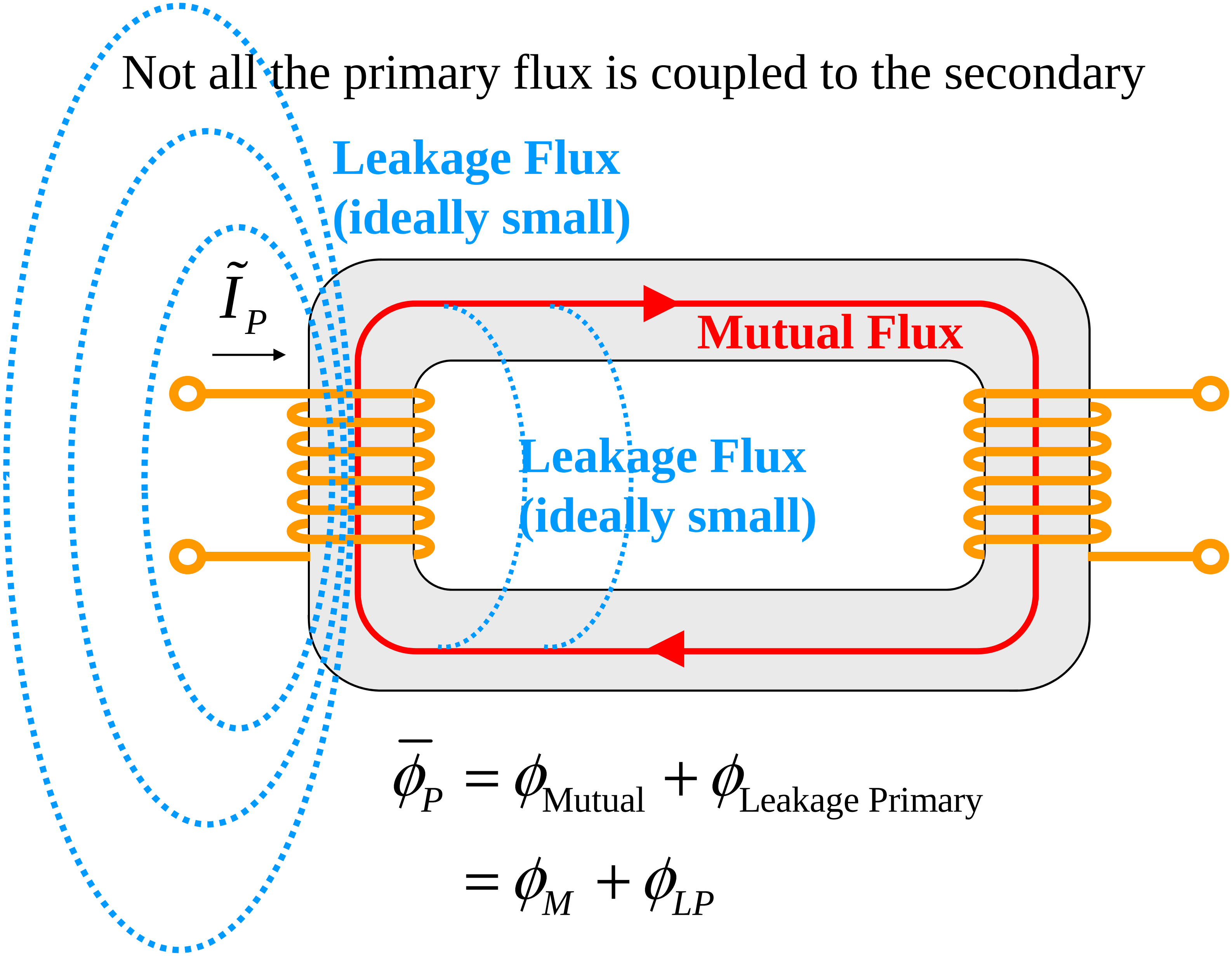
How does this flux effect the secondary?

Flux Linkage – When a voltage is applied across the primary of a transformer the average flux in the winding will be:

$$\bar{\phi} = \frac{1}{N_P}\int v_P\left(t\right)dt$$

How does this flux effect the secondary?

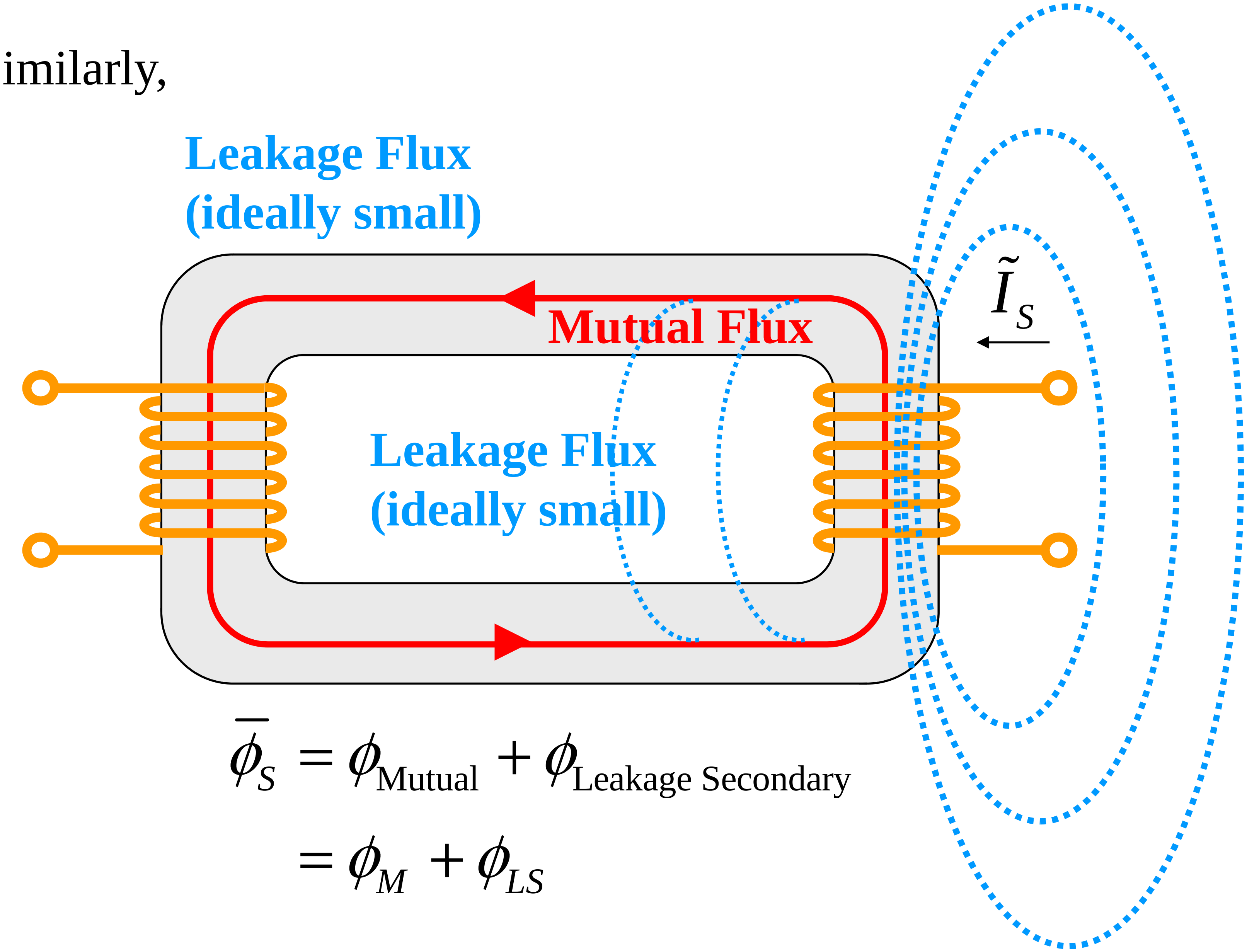
Not all the primary flux is coupled to the secondary



$$\bar{\phi}_P = \phi_{\text{Mutual}} + \phi_{\text{Leakage Primary}}$$

$$= \phi_M + \phi_{LP}$$

Similarly,



$$\bar{\phi}_S = \phi_{\text{Mutual}} + \phi_{\text{Leakage Secondary}}$$

$$= \phi_M + \phi_{LS}$$

Re-express Faraday's Law as:

$$v_P\left(t\right) = N_P \frac{d\overline{\phi}_P}{dt}$$

$$= N_P \frac{d\phi_M}{dt} + N_P \frac{d\phi_{LP}}{dt}$$

$$= e_P\left(t\right) + e_{LP}\left(t\right)$$

Similarly:

$$v_S\left(t\right) = N_S \frac{d\overline{\phi}_S}{dt}$$

$$= N_S \frac{d\phi_M}{dt} + N_S \frac{d\phi_{LS}}{dt}$$

$$= e_S\left(t\right) + e_{LS}\left(t\right)$$

The primary voltage due to the *mutual flux* is:

$$e_P(t) = N_P \frac{d\phi_M}{dt}$$

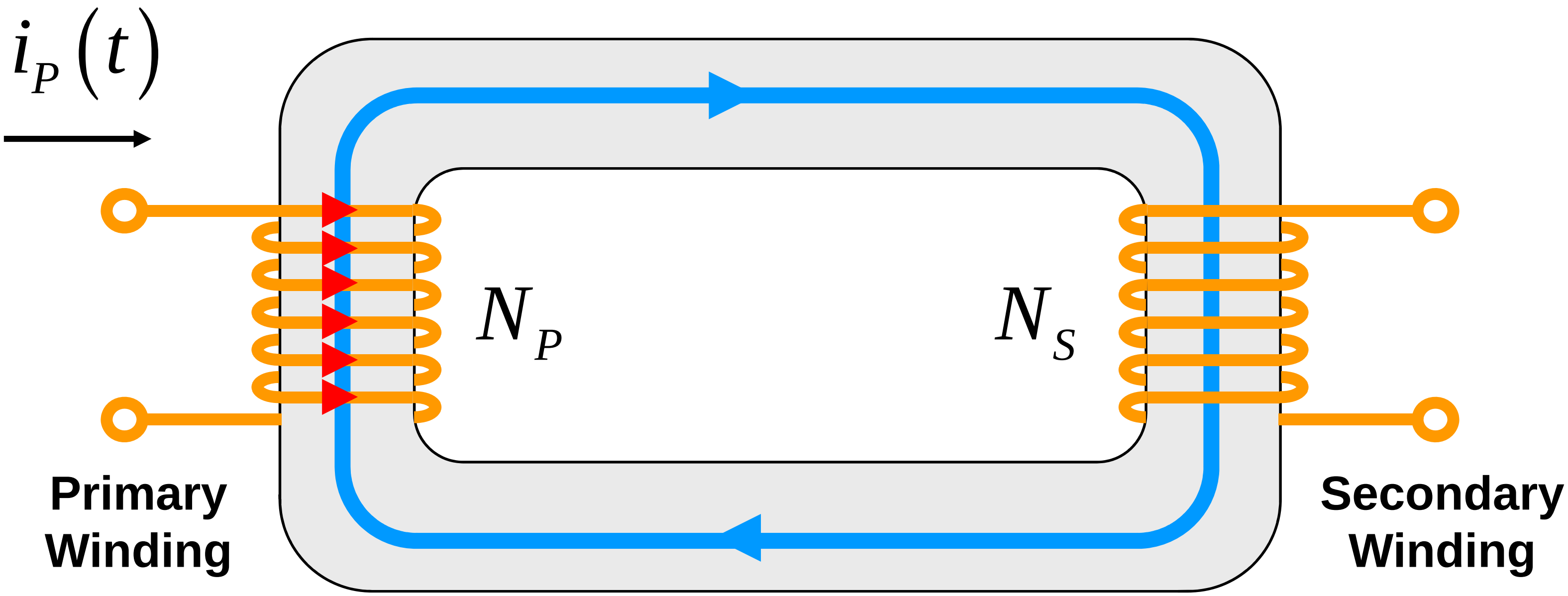
The secondary voltage due to the *mutual flux* is:

$$e_S(t) = N_S \frac{d\phi_M}{dt}$$

Hence:

$$\frac{e_P(t)}{N_P} = \frac{d\phi_M}{dt} = \frac{e_S(t)}{N_S} \Rightarrow \quad \frac{e_P(t)}{e_S(t)} = \frac{N_P}{N_S} \approx \frac{v_P(t)}{v_S(t)}$$

Magnetization Current in a Real Transformer



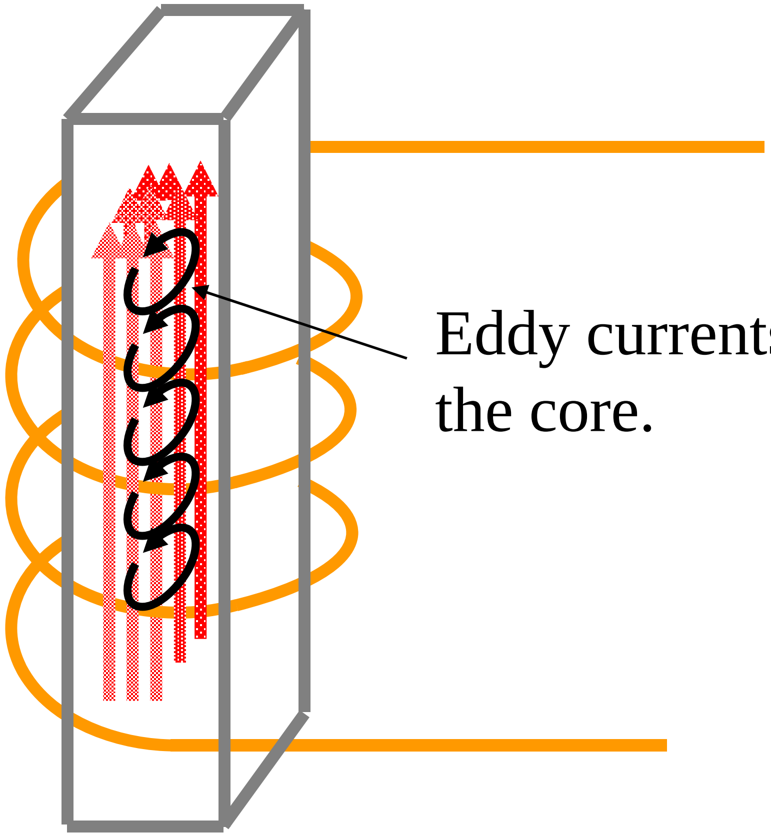
Note again that when a primary source is connected, current flows in the primary even the secondary is open circuited. This is the current required to produce the flux in the (real) ferromagnetic core.

Magnetization Current in a Real Transformer

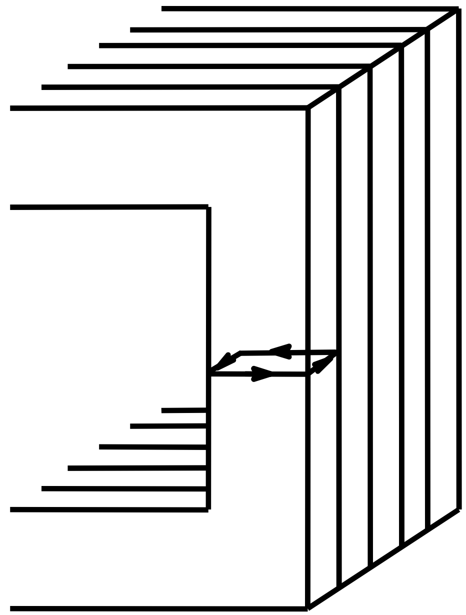
The applied current consists of two components:

- *Magnetization current*; the current required to produce the flux in the core, and
- The core loss current to account for eddy current and hysteresis loss.

Eddy Current Loss – A changing flux will induce a current in the magnetic core just as is does for a wire.



Since the eddy current loss is proportional to the path length, making the core using the laminated, layered structure shown ***minimizes*** these losses.



Magnetization Current: Ignoring leakage, recall from Slide 28 that the average flux in the core is:

$$\bar{\phi} = \frac{1}{N_P} \int v_P(t)\, dt$$

If the primary voltage is:

$$v_P(t) = V_M \cos \omega t$$

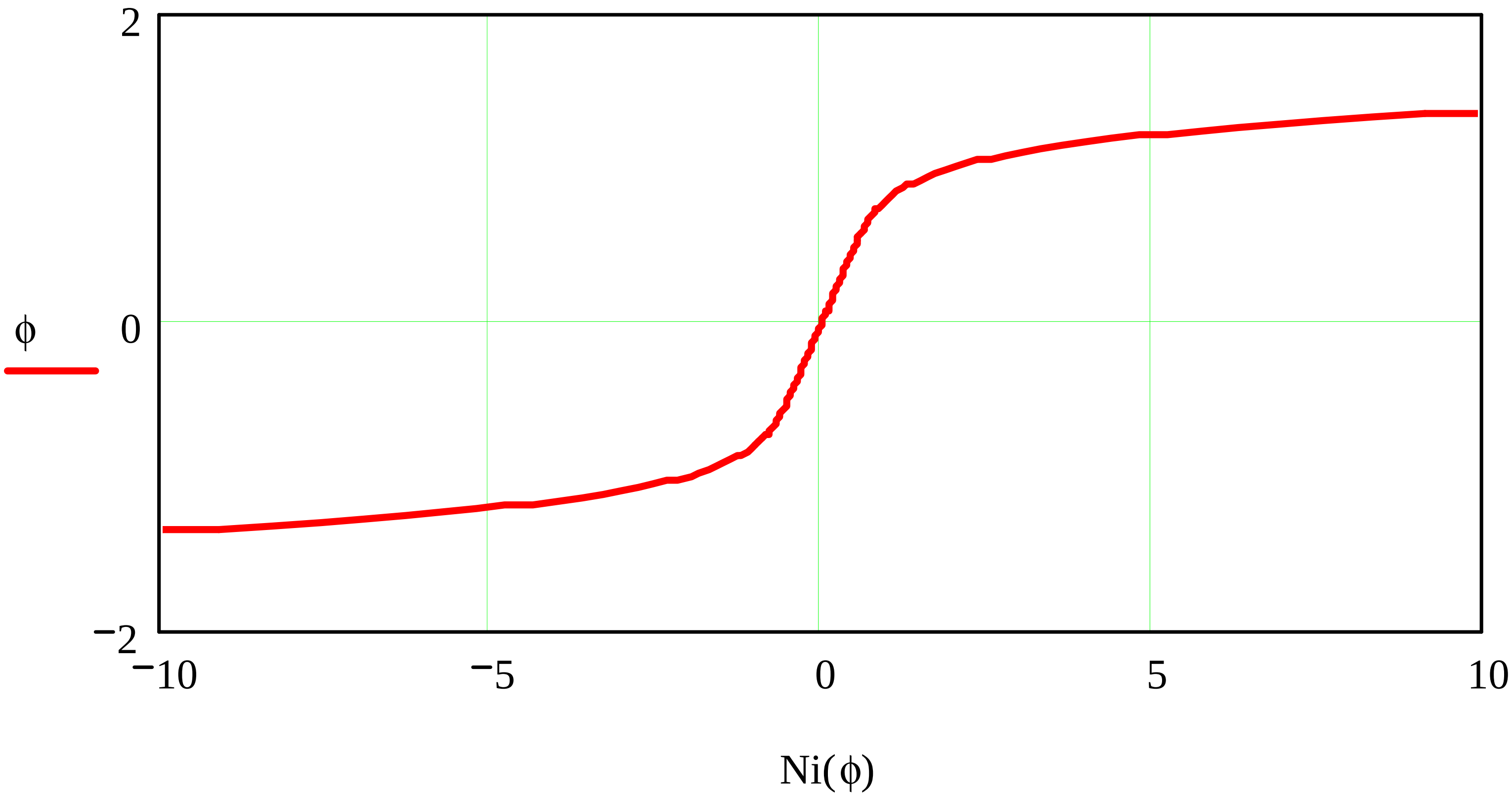
then

$$\bar{\phi} = \frac{V_M}{\omega N_P} \sin \omega t$$

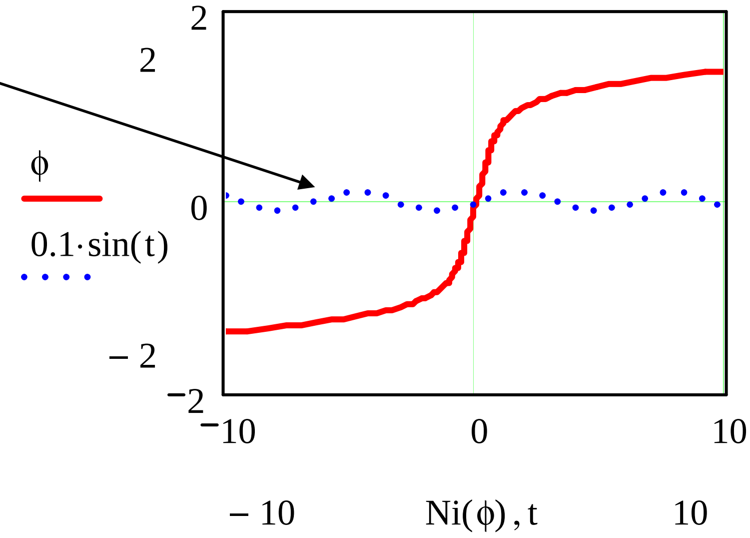
Note that the flux (and hence the magnetization current) lags the applied voltage by 90°.

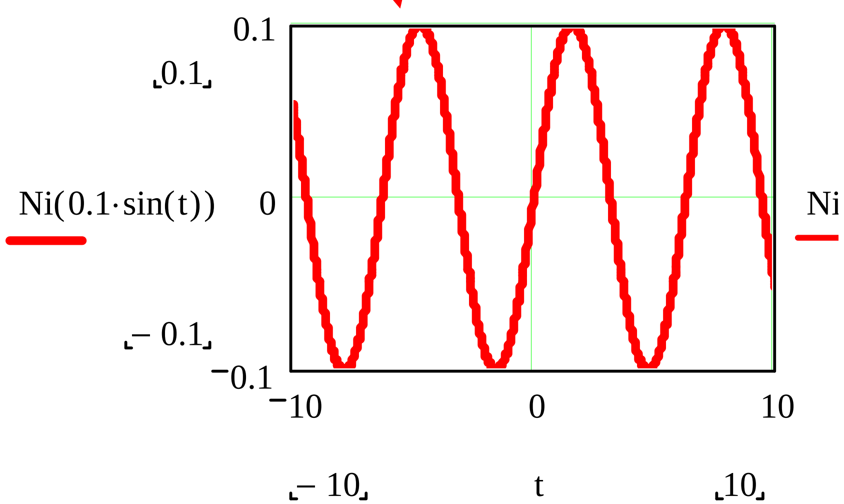
Example – Suppose the magnetization curve for a real core can be modeled as

$$Ni(\phi) := \phi^7 + \phi$$

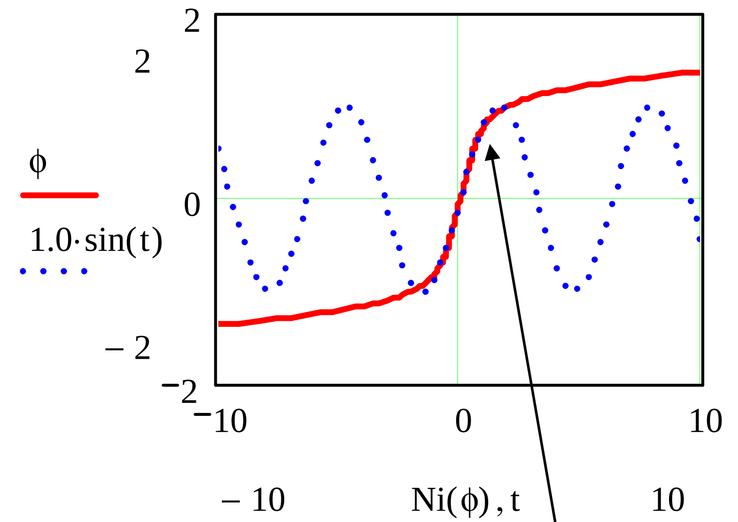


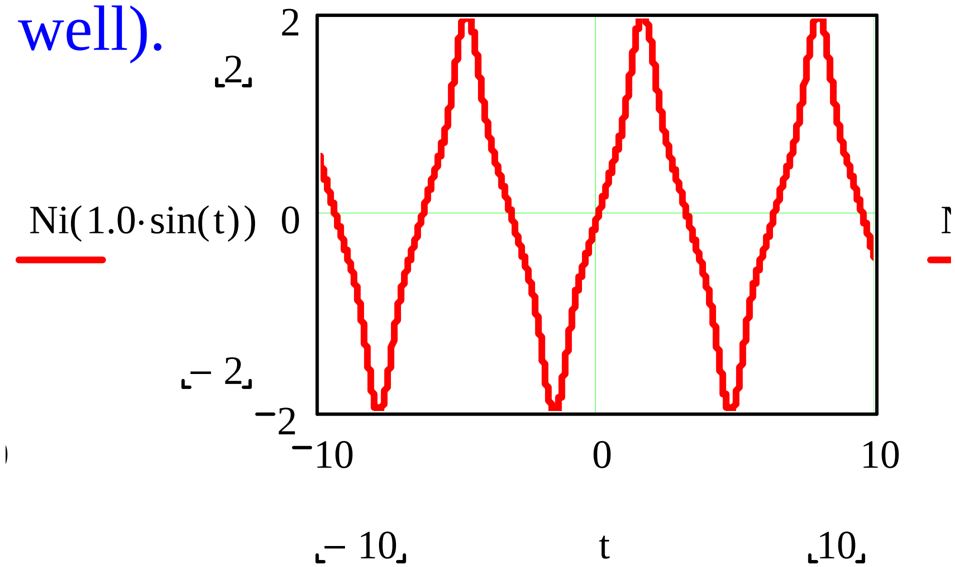
If the sinusoidal flux variation is ‘small’, the magnetization current is nearly sinusoidal as well.

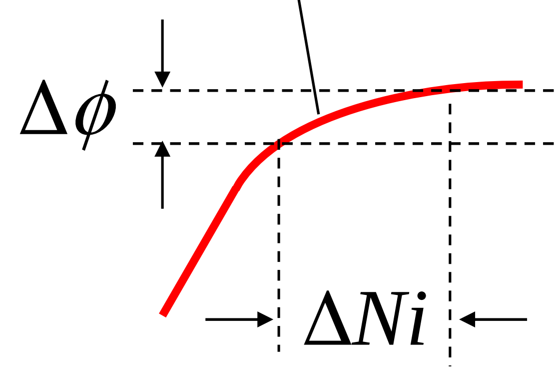


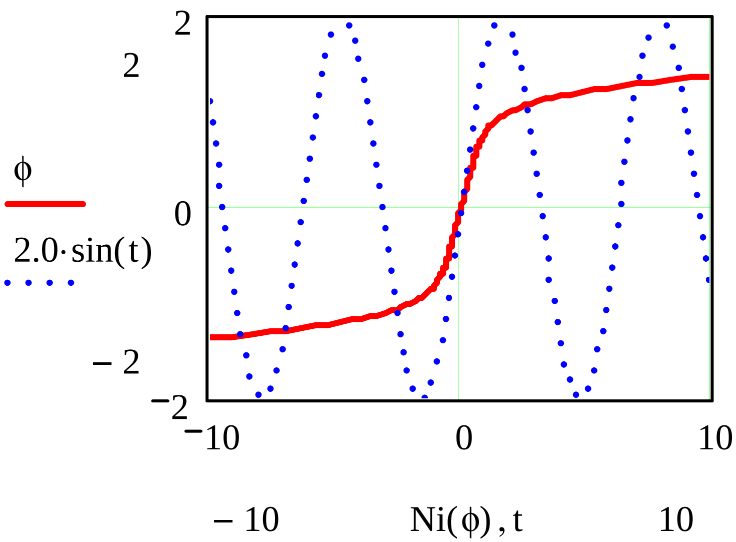


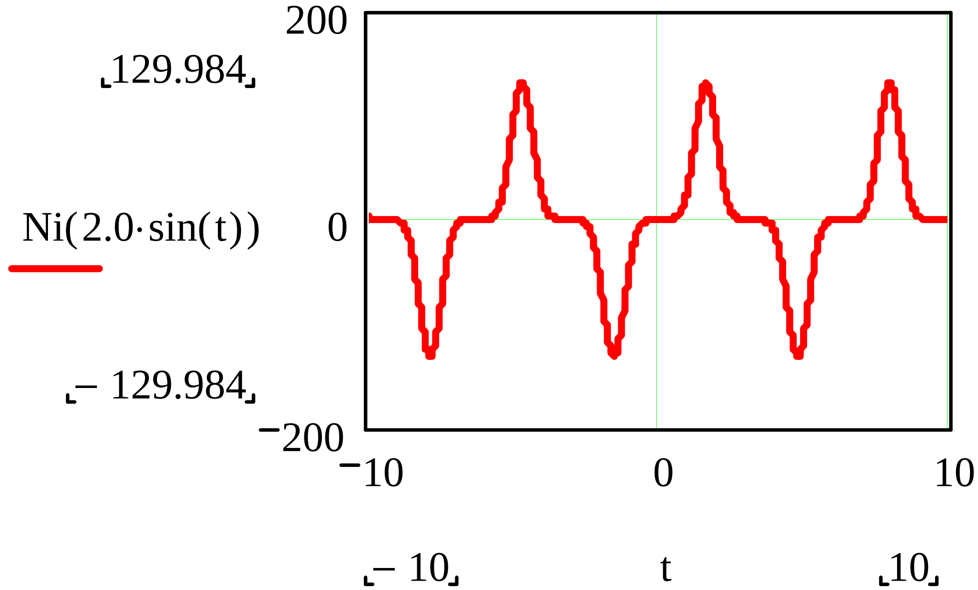
Once the peak flux reaches the saturation point in the core, the magnetization current is clearly on longer sinusiodal. Note also that even a small additional increase in the flux will cause a large corresponding increase in the magnetization current. (See next slide as well).








2
2
0
− 2
−2
−10
0
10
ϕ
2.0·sin(t)
− 10
Ni(ϕ) , t
10


200
129.984
Ni(2.0·sin(t))
0
− 129.984
−200
−10
0
10
− 10
t
10

Hysteresis and Eddy Current Loss – The other component of the no-load current is current required to supply power for the hysteresis and eddy current losses. These are core losses.

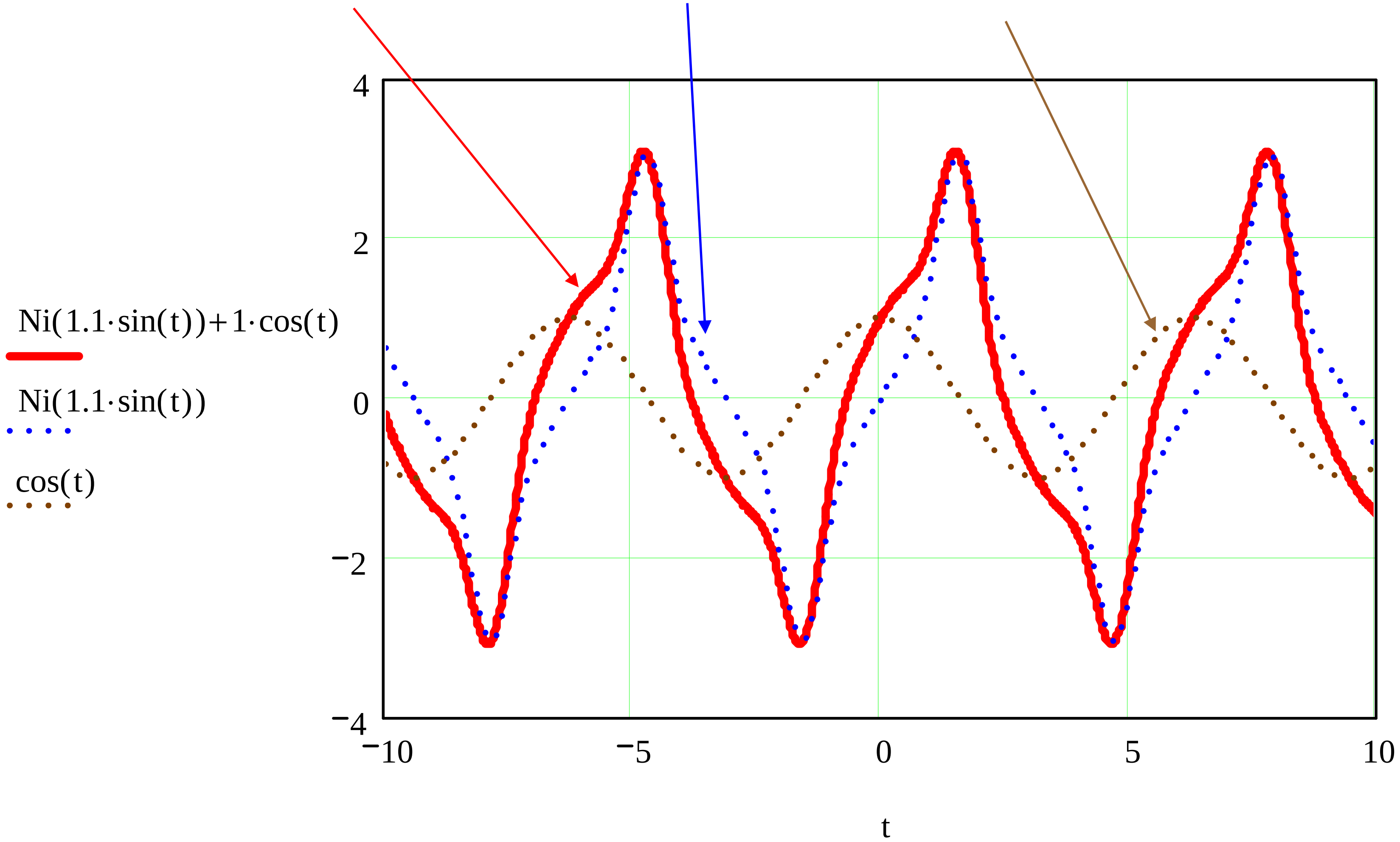
As before, assume that the flux in the core is sinusoidal. By Faraday's law, the eddy currents in the core will be proportional to

$$i_{eddy} \sim \frac{d\phi}{dt}$$

The eddy currents will be largest when the core flux is passing through zero.

Since core losses are resistive in nature, the eddy current will be in-phase with the applied primary voltage.

$$i_{excitation} = i_{magnetization} + i_{hysteresis+eddy}$$



Linear Circuit Model for a Real Transformer

How do we model a *linear* real transformer?

Copper losses
Eddy current losses
Hysteresis losses
Leakage flux

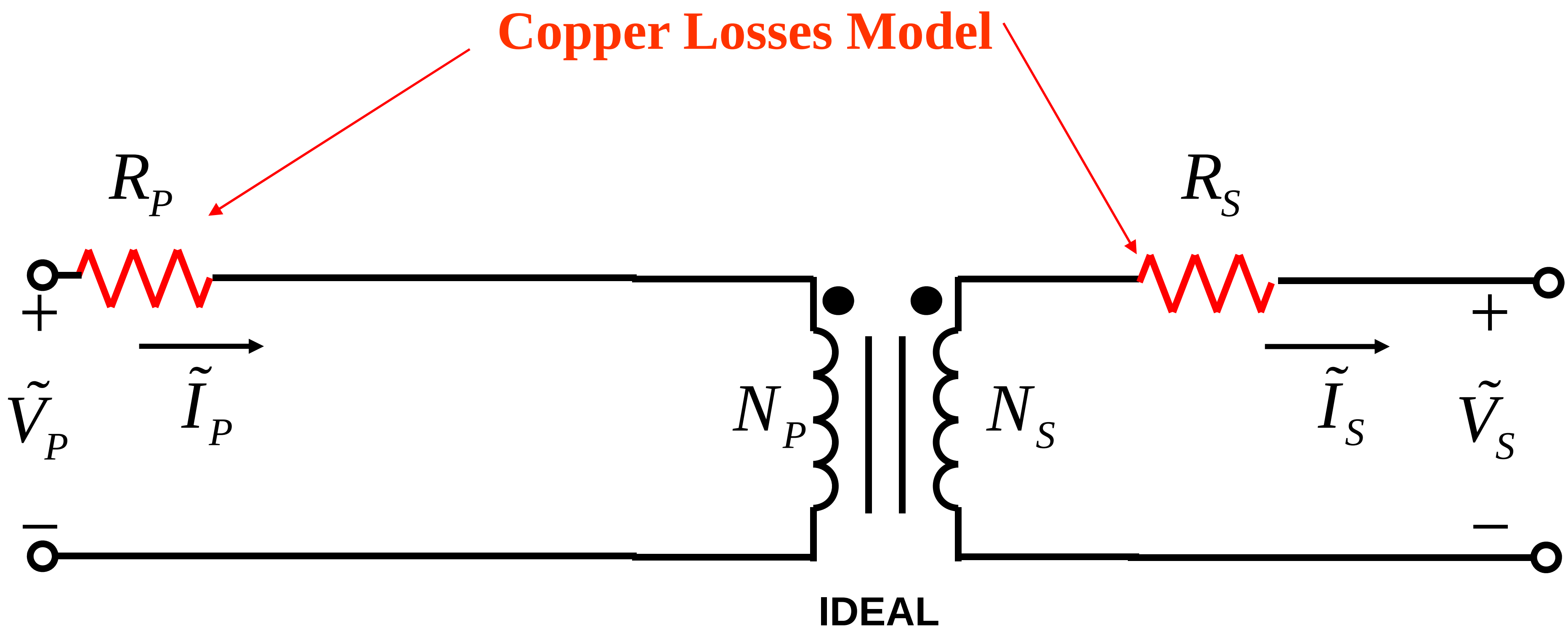
Not the magnetic saturation effects – these are non linear.

Linear Circuit Model for a Real Transformer

How do we model a real transformer?

Copper losses – Add a resistance (R_P and R_S) in the primary and secondary circuit, respectively.

Linear Circuit Model for a Real Transformer



Linear Circuit Model for a Real Transformer

How do we model a real transformer?

Leakage flux

Leakage Flux

Recall from Slide 31 that,

$$e_{LP}(t) = N_P \frac{d\phi_{LP}}{dt} \qquad e_{LS}(t) = N_S \frac{d\phi_{LS}}{dt}$$

Since much (most) of the leakage flux path is through air, and since air has a constant reluctance which is much greater than that of the core, the primary/secondary leakage flux is proportional to the primary/secondary coil current, respectively, or

$$\phi_{LP} = \frac{N_P i_P}{\mathscr{R}} = \mathscr{P}\, N_P i_P$$

$$\phi_{LS} = \frac{N_S i_S}{\mathscr{R}} = \mathscr{P}\, N_S i_S$$

Reluctance of flux path

Permeance of flux path

Leakage Flux

$$e_{LP}(t) = N_P \frac{d\phi_{LP}}{dt} = N_P{}^2 \mathscr{P} \frac{di_P}{dt} = L_P \frac{di_P}{dt}$$

$$e_{LS}(t) = N_S \frac{d\phi_{LS}}{dt} = N_S{}^2 \mathscr{P} \frac{di_S}{dt} = L_S \frac{di_S}{dt}$$

Recall Slide 31,

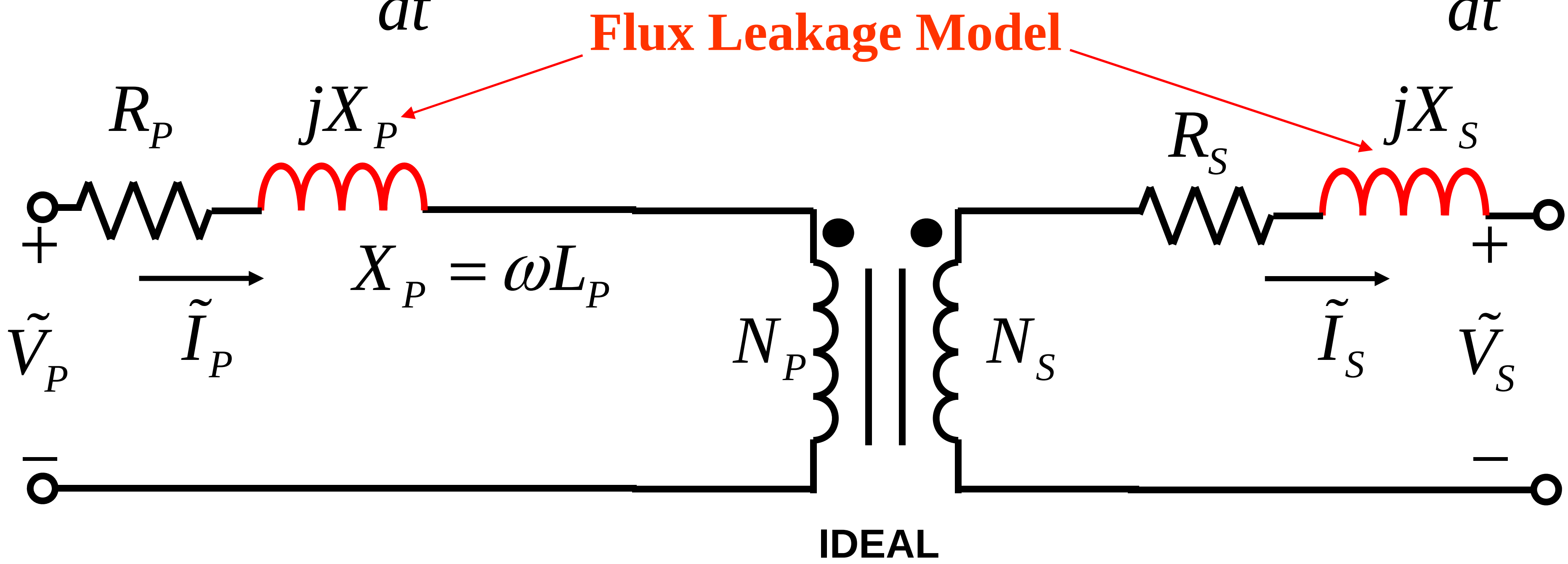
$$v_P(t) = e_{LP}(t) + e_P(t) = N_P \frac{d\phi_{LP}}{dt} + N_P \frac{d\phi_M}{dt}$$

$$\Rightarrow e_P(t) = N_P \frac{d\phi_M}{dt}$$

which suggests an inductor as a model.

$$e_{LP}(t) = L_P \frac{di_P}{dt}$$

$$e_{LS}(t) = L_S \frac{di_S}{dt}$$



$$v_P(t) = e_{LP}(t) + e_P(t) = N_P \frac{d\phi_{LP}}{dt} + N_P \frac{d\phi_M}{dt}$$

How do we model these effects?

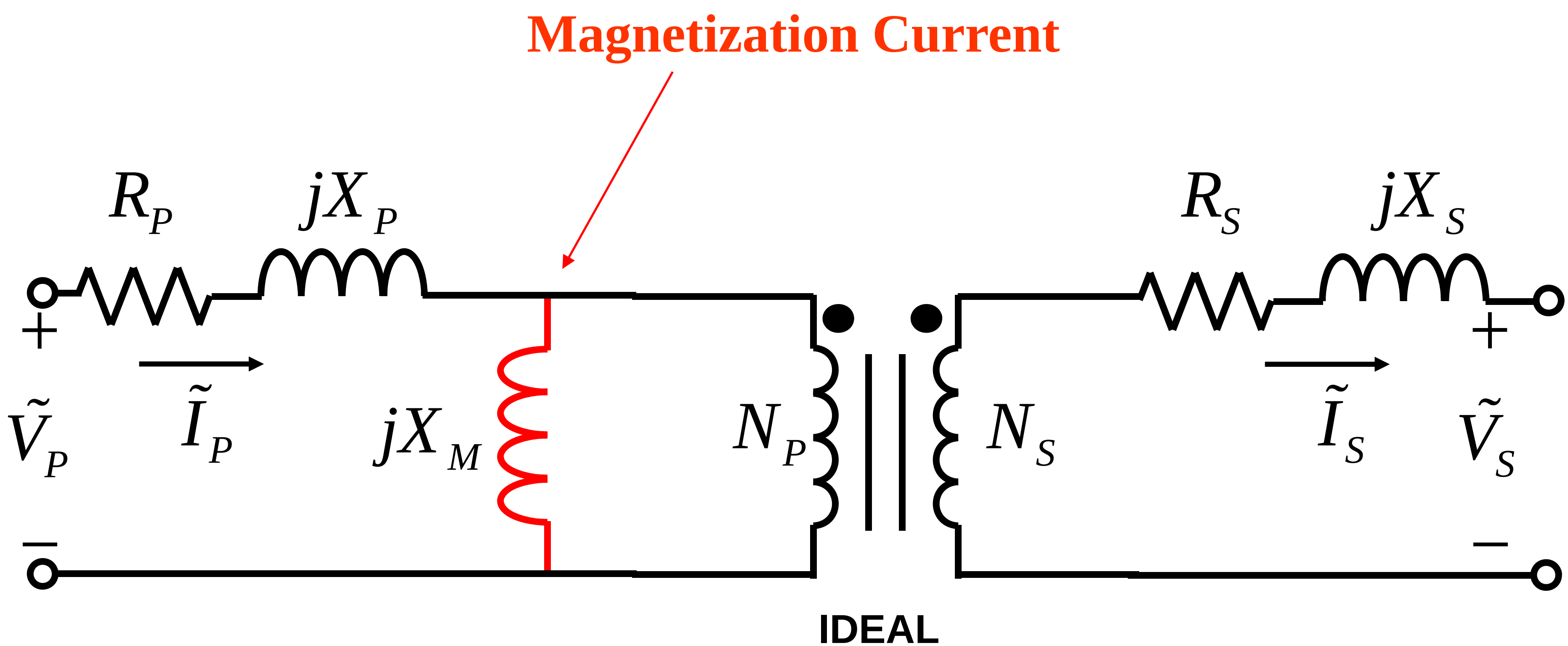
- Magnetization current
- Hysteresis (core) losses

Recall (Slide 37) that the magnetization current (*in the unsaturated region*) is proportional to the voltage applied to the core and lags the applied voltage by 90°,

$$v_P(t) = V_M \cos\omega t \qquad \bar{\phi} = \frac{V_M}{\omega N_P}\sin\omega t \qquad \bar{\phi} = \frac{N_P i_P}{\mathscr{R}}$$

$$\bar{\phi} = \frac{1}{N_P}\int v_P(t)\,dt \Rightarrow v_P(t) = N_P \frac{d\bar{\phi}}{dt}$$

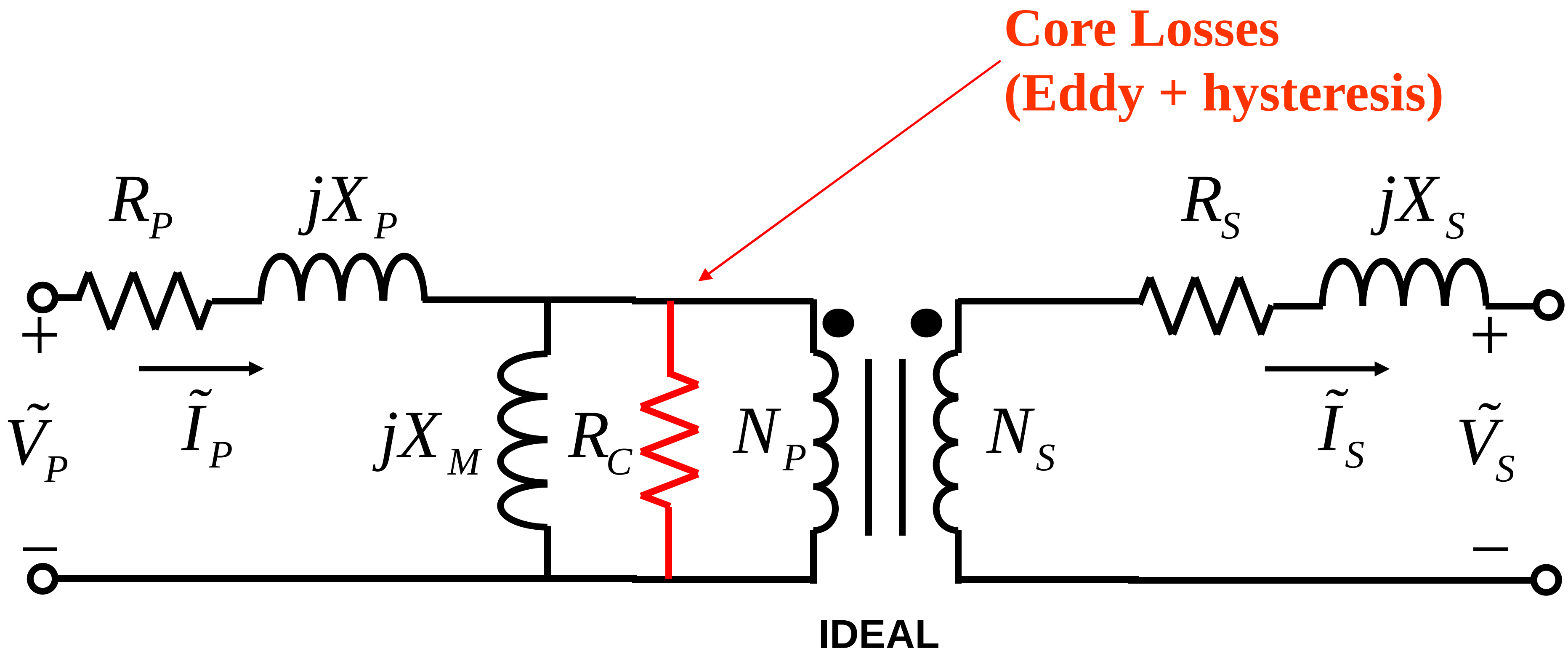
Hence it can be modeled by a reactance (an inductor) connected across the primary voltage source.



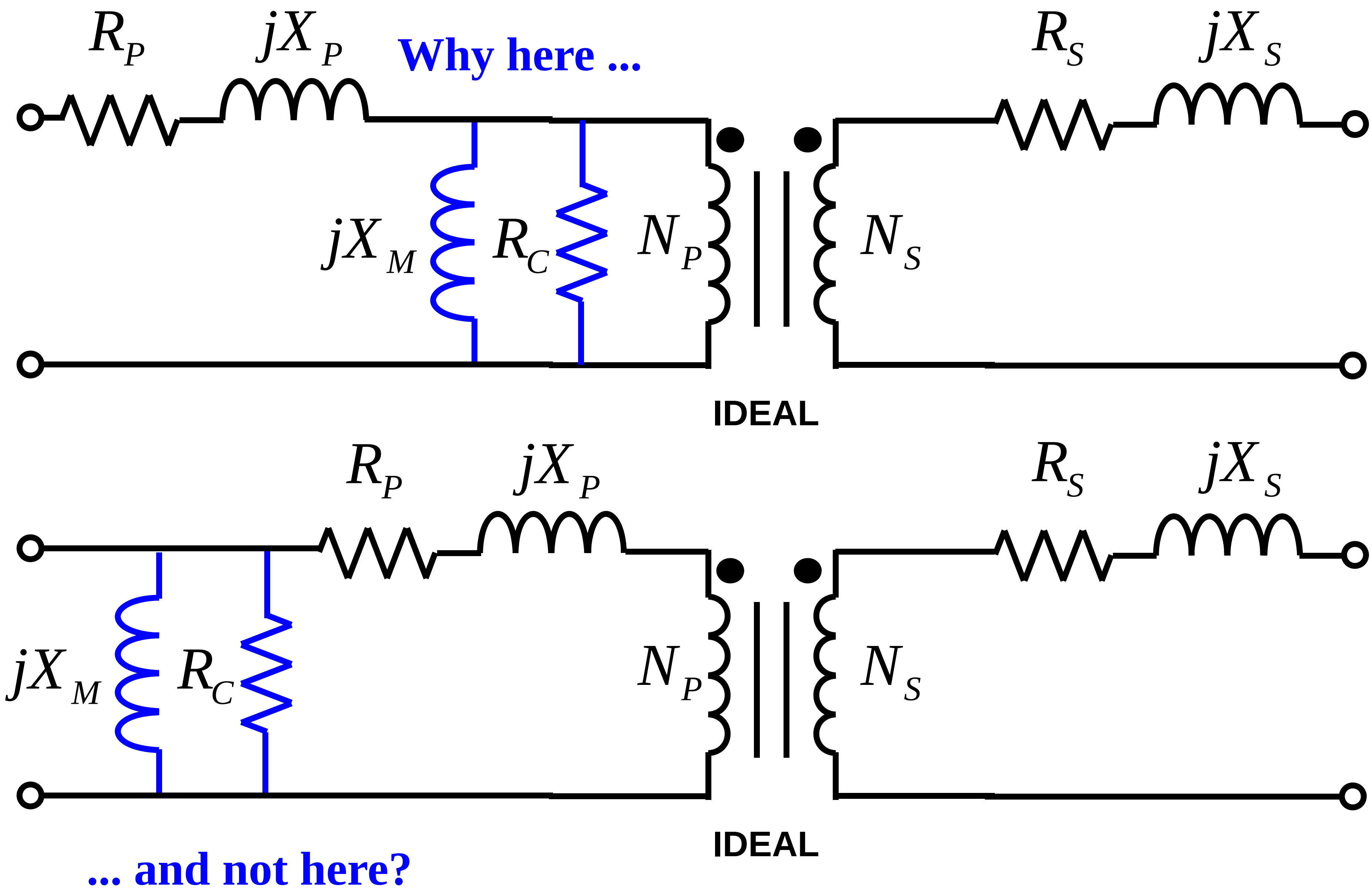
$$v_P(t) = e_{LP}(t) + e_P(t) = N_P \frac{d\phi_{LP}}{dt} + N_P \frac{d\phi_M}{dt}$$

The core loss current $i_{hysteresis + eddy}$ is a current proportional to the voltage applied to the core that is in-phase with the applied voltage, hence it can be modeled by a resistance R_C connected across the primary voltage source.

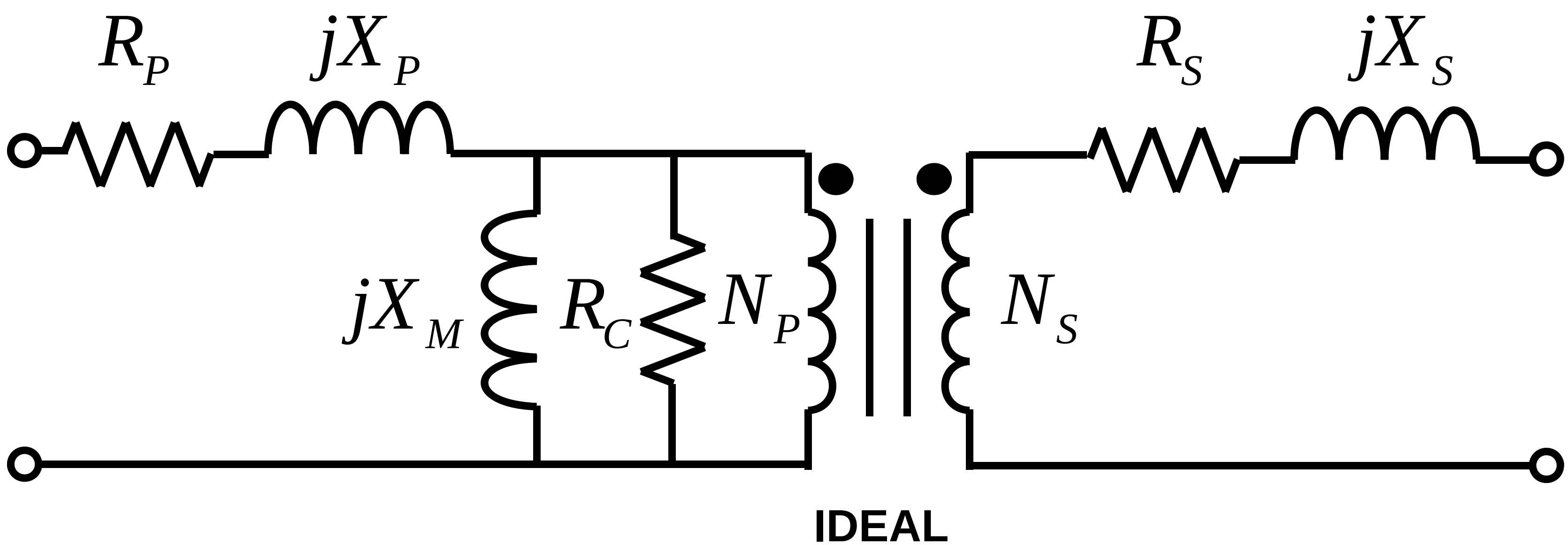
Remember, the core effects are really nonlinear, and the circuit model we have produced provides only a linear approximation in this complicated situation.



Core Losses
(Eddy + hysteresis)
R_P
jX_P
R_S
jX_S
+
$\tilde{V}_P$
$\tilde{I}_P$
jX_M
R_C
N_P
N_S
$\tilde{I}_S$
$\tilde{V}_S$
−
IDEAL


R_P
jX_P
Why here ...
R_S
jX_S
jX_M
R_C
N_P
N_S
IDEAL
R_P
jX_P
R_S
jX_S
jX_M
R_C
N_P
N_S
IDEAL
... and not here?

Because the voltage actually applied to the core is equal to the applied voltage less the internal voltage drops of the windings.



Simplifications of the Model

Recall from Slide 8: $$\frac{v_P(t)}{v_S(t)} = \frac{N_p}{N_S} = Turns\ Ratio$$

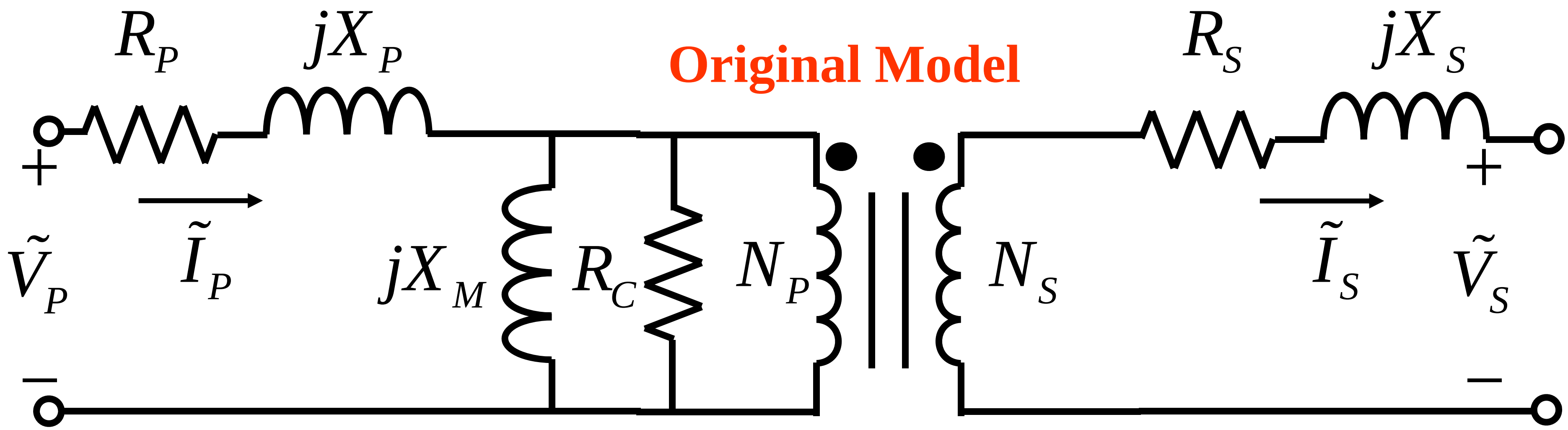
So that

$$\frac{\tilde{V}_P}{\tilde{V}_S} = \frac{N_p}{N_S} = a$$

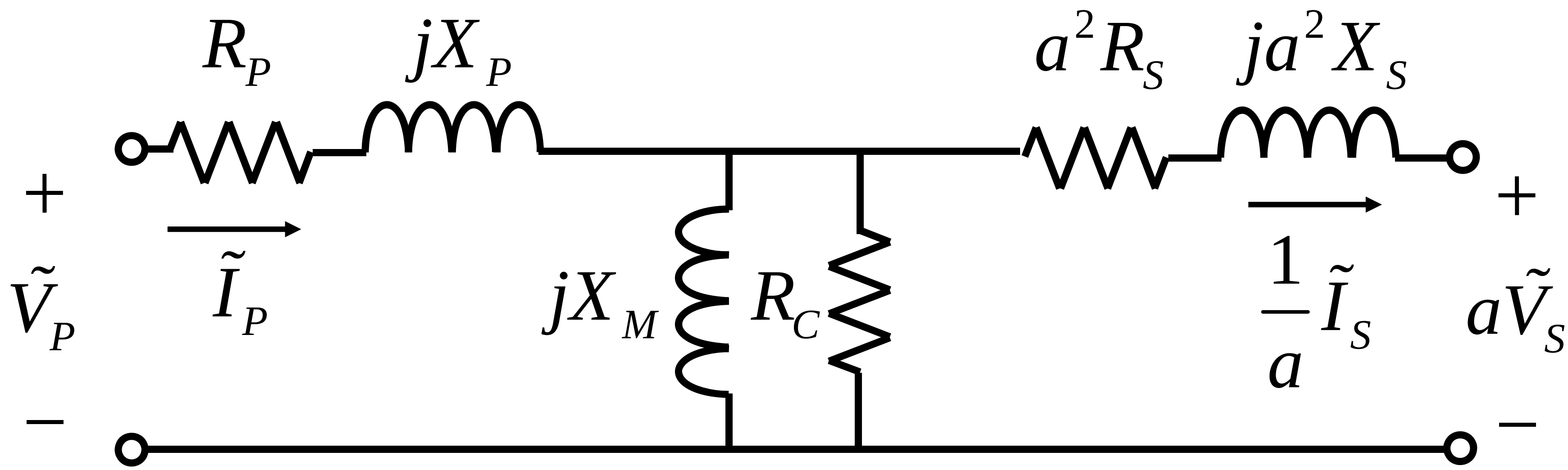
Also recall from Slide 22:

$$Z_{in} = \left(\frac{N_p}{N_S}\right)^2 Z_L = a^2 Z_L$$

Simplifications of the Model

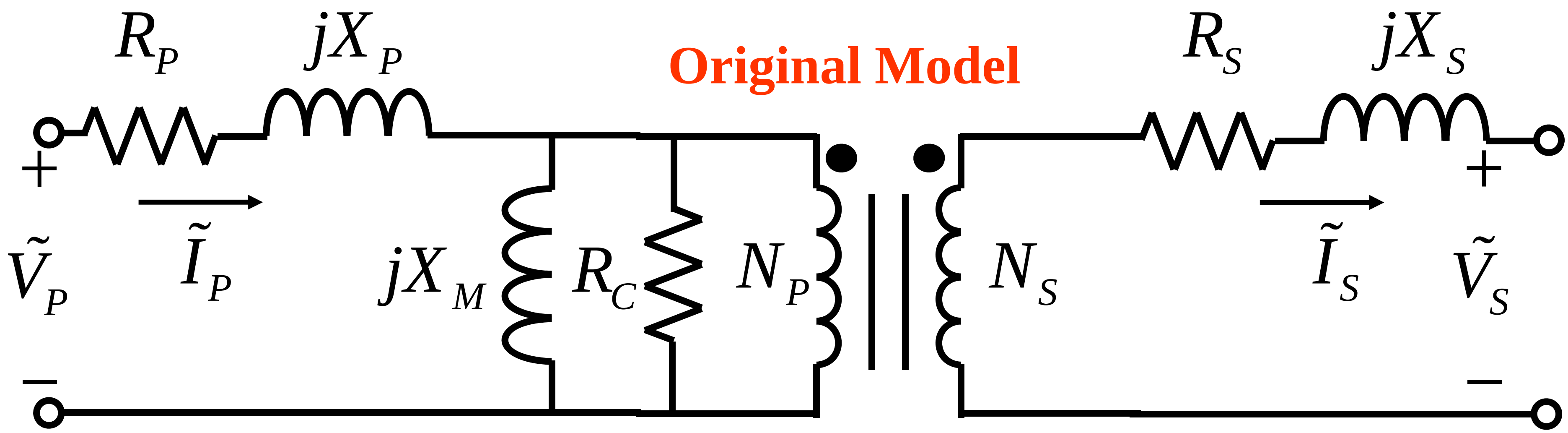


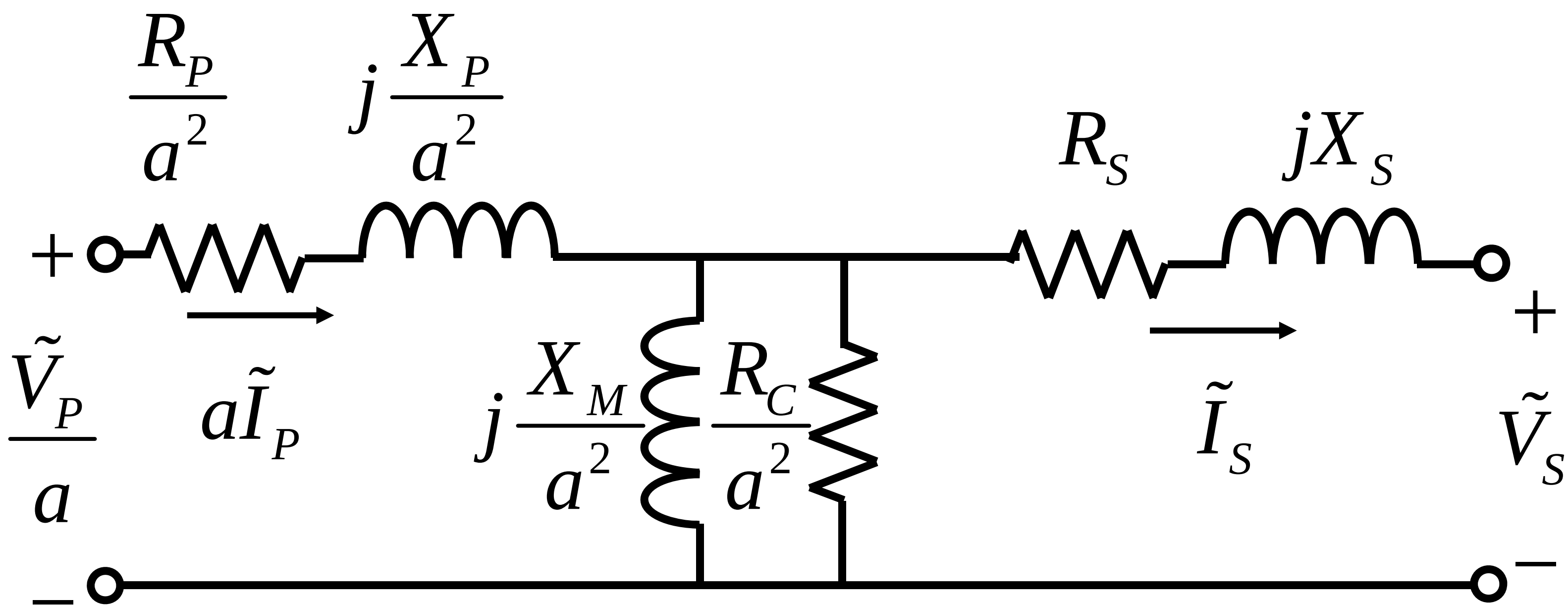
Original Model



Model referred to its primary side

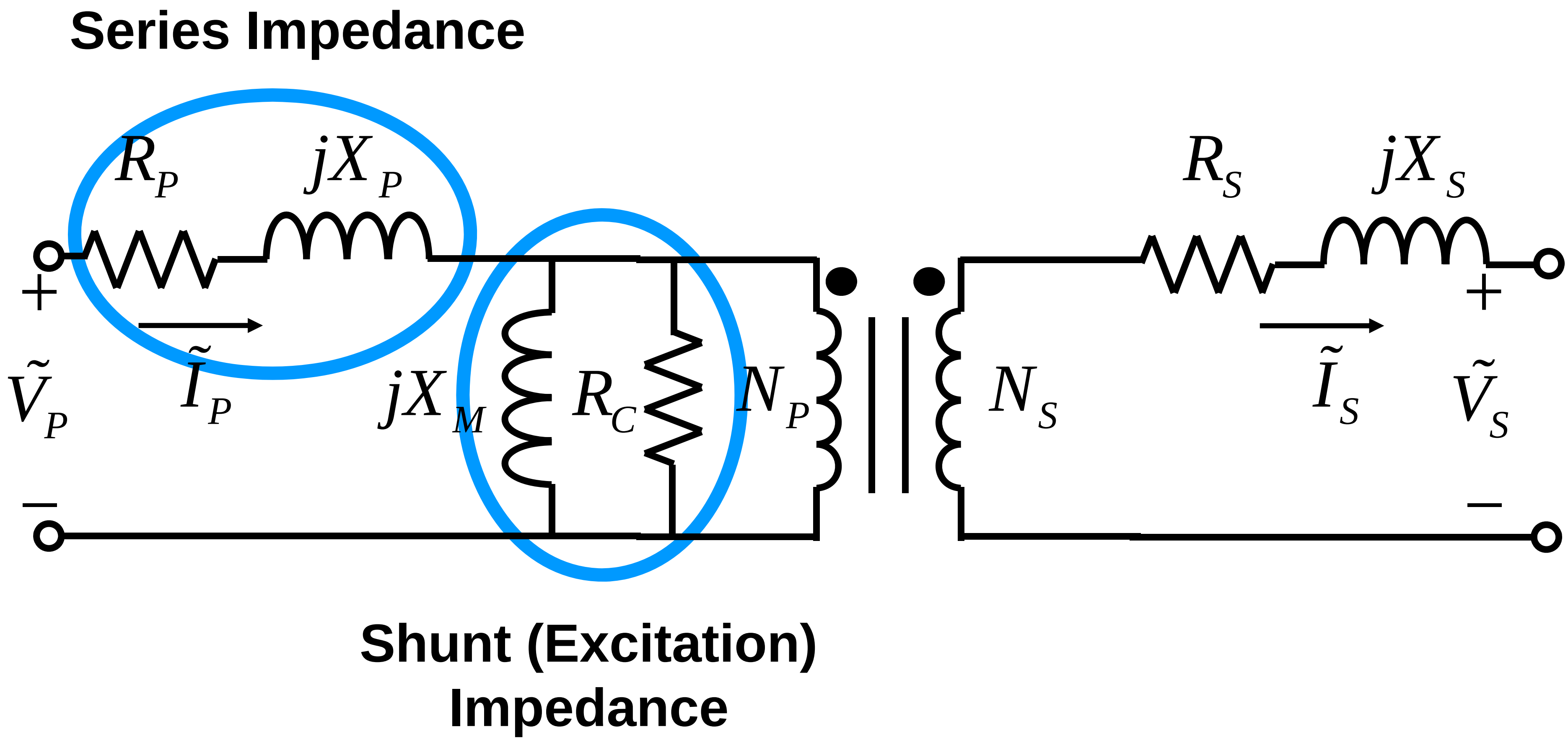
Simplifications of the Model





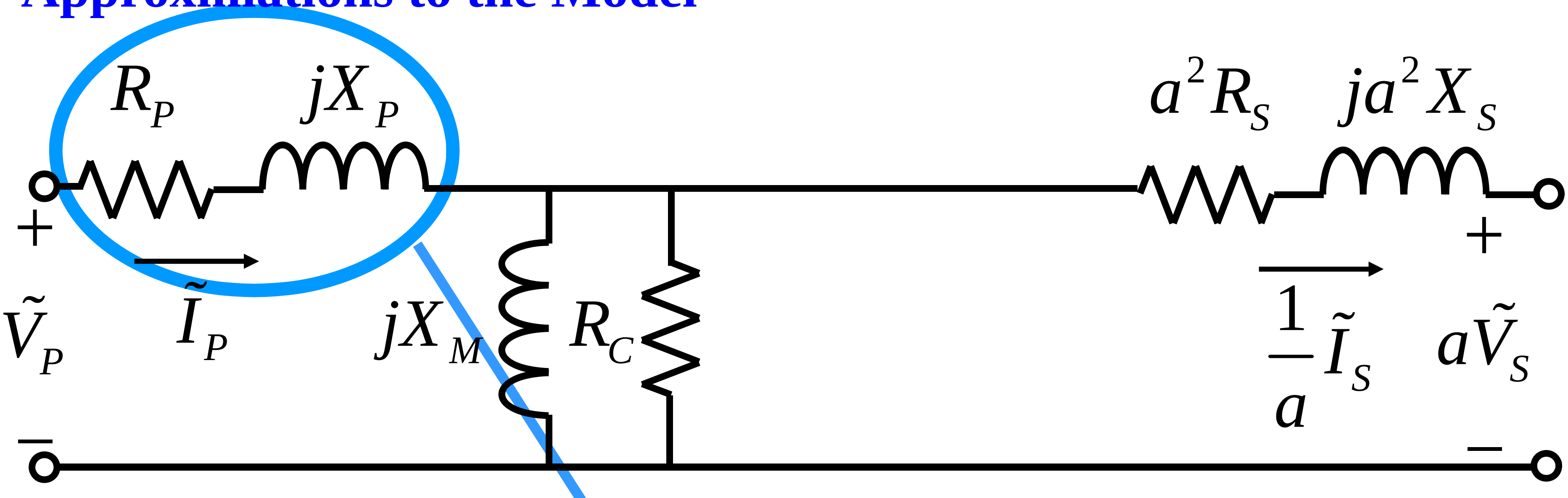
Model referred to its secondary side

Approximations to the Model

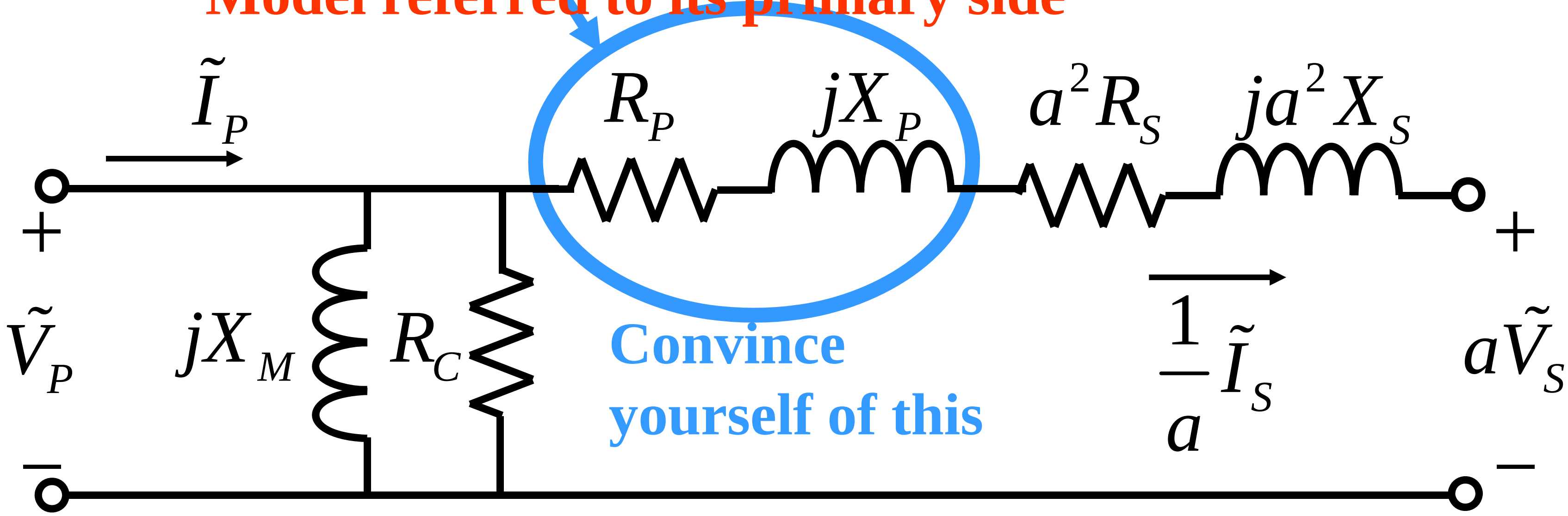


For large transformers (several kilovoltamperes or more), the shunt impedance is much greater than the series impedance, and some approximations can be made. For example...

Approximations to the Model

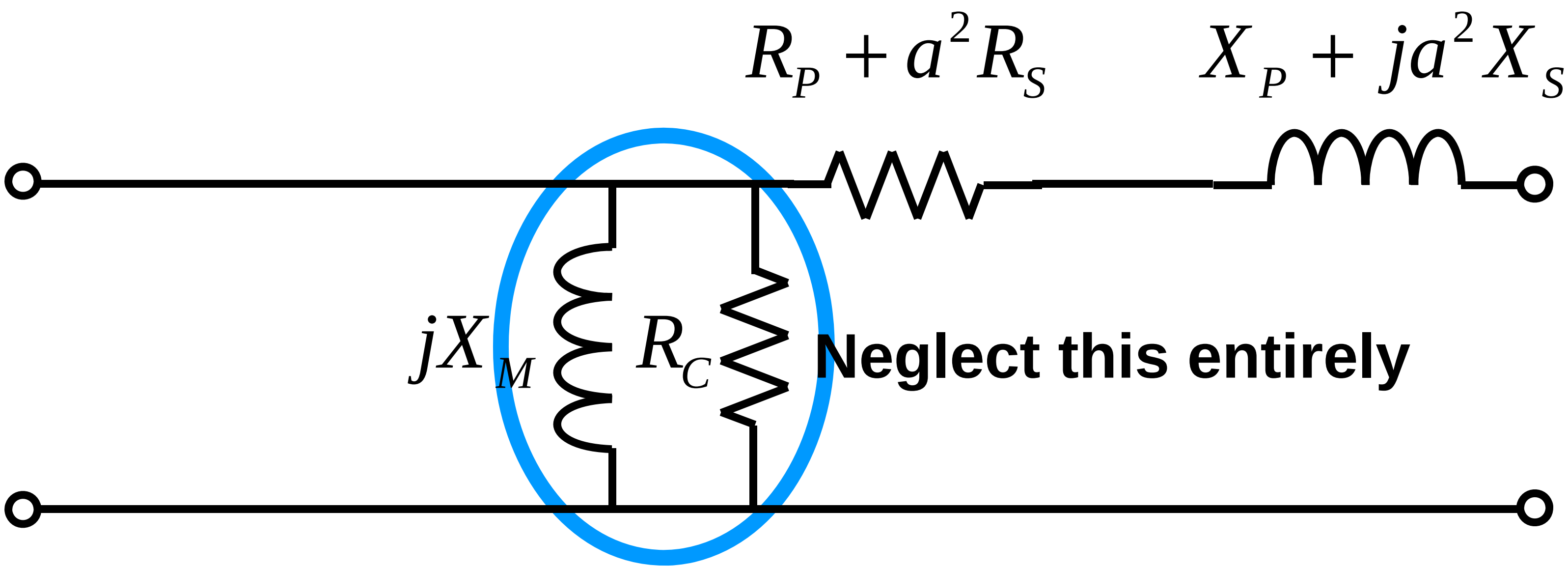


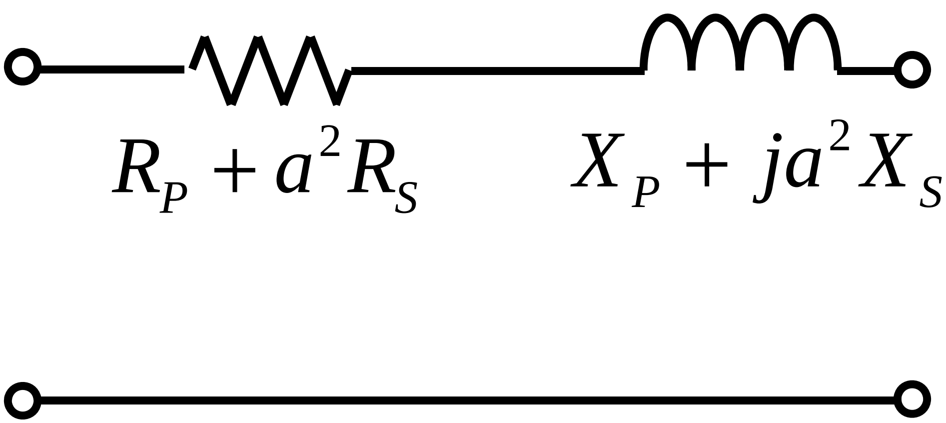
Model referred to its primary side



Approximation

Further Approximations to the Model



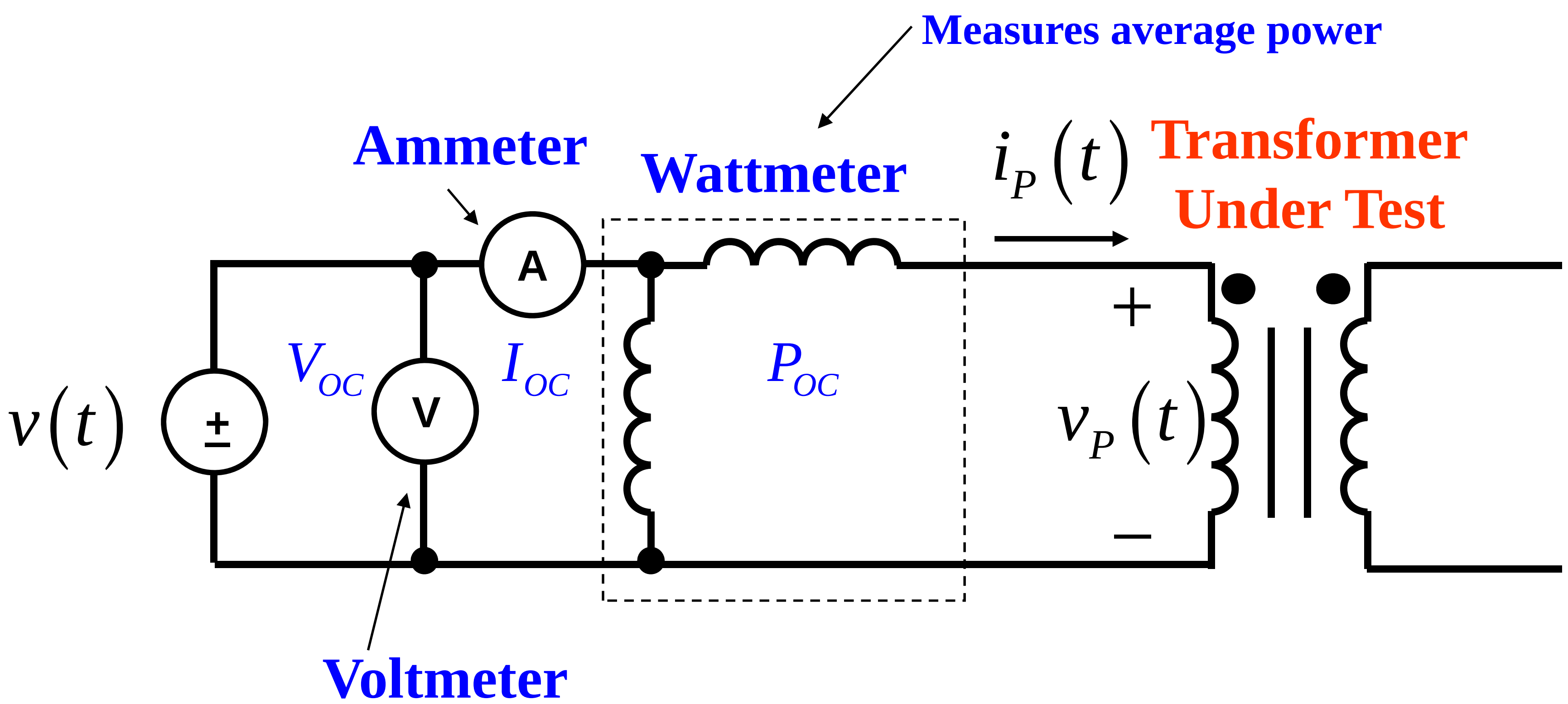


Determination of Values of Components in the Model
(EEL 4201, Lab 6)

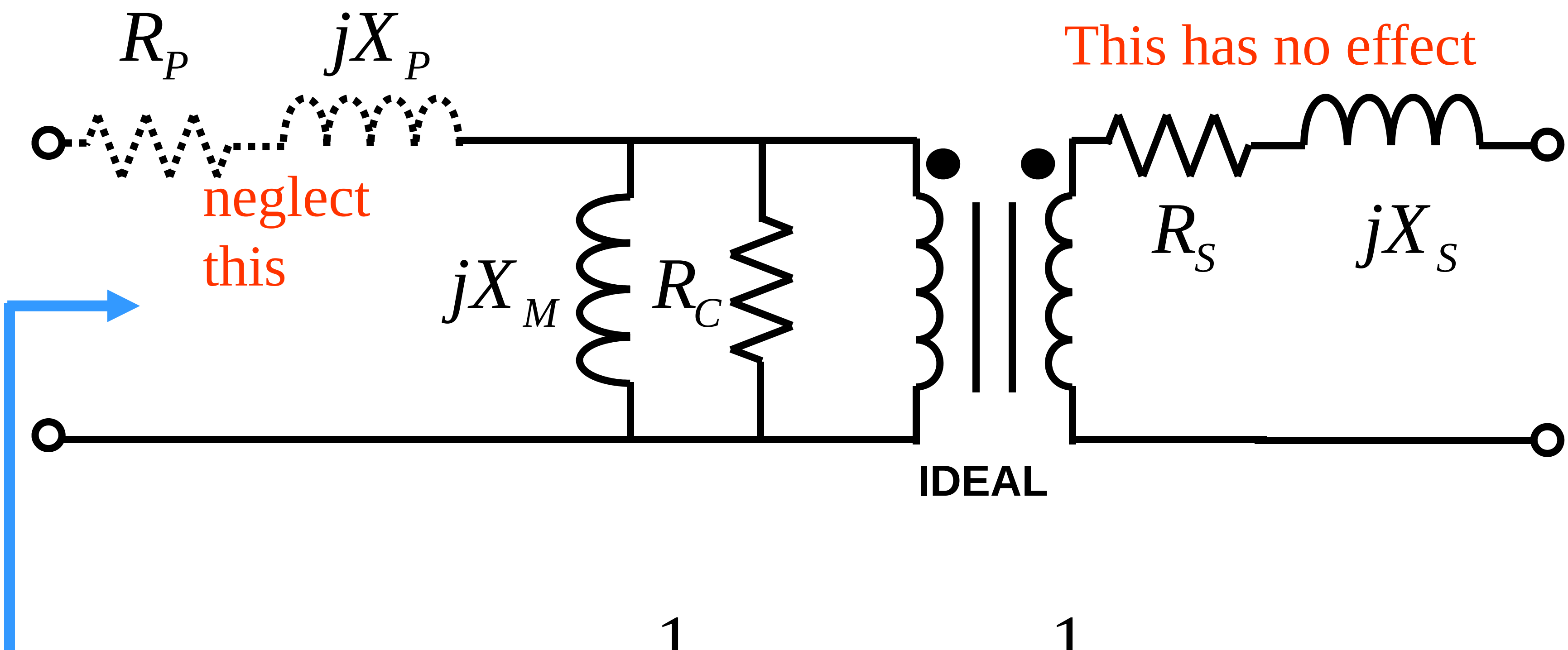
Only two tests required:

- **Open Circuit Test**
- **Short Circuit Test**

Open Circuit Test



Open Circuit Test – Since the magnetizing impedance is generally much greater that the series primary impedance, the series impedance can be neglected.



$$Z_E = R_P + jX_P + \frac{1}{\frac{1}{jX_M} + \frac{1}{R_C}} \approx \frac{1}{\frac{1}{jX_M} + \frac{1}{R_C}}$$

Open Circuit Test

$$Z_E = \frac{1}{\frac{1}{R_C} + \frac{1}{jX_M}}, Y_E = \frac{1}{Z_E} = \frac{1}{R_C} - j\frac{1}{X_M}$$

$$|Y_E| = \frac{I_{OC}}{V_{OC}}, \quad PF = \cos\theta = \frac{P_{OC}}{V_{OC} I_{OC}}$$

$$Y_E = \frac{I_{OC}}{V_{OC}} \angle -\theta = \frac{I_{OC}}{V_{OC}} \angle - \cos^{-1}\left(PF\right)$$

The power factor is always lagging for a real transformer

Open Circuit Test

$$Y_E = \frac{1}{Z_E} = \frac{1}{R_C} - j\frac{1}{X_M}$$

$$|Y_E| = \sqrt{\left(\frac{1}{R_C}\right)^2 + \left(\frac{1}{X_M}\right)^2}$$

$$\theta = \tan^{-1}\left(\frac{\frac{1}{X_M}}{\frac{1}{R_C}}\right) = \tan^{-1}\left(\frac{R_C}{X_M}\right)$$

$$\theta = \cos^{-1} PF = \tan^{-1}\left(\frac{R_C}{X_M}\right)$$

$$R_C = X_M \tan\left[\cos^{-1} PF\right]$$

$$|Y_E| = \sqrt{\left(\frac{1}{R_C}\right)^2 + \left(\frac{1}{X_M}\right)^2}$$

$$= \sqrt{\left(\frac{1}{X_M \tan\left[\cos^{-1} PF\right]}\right)^2 + \left(\frac{1}{X_M}\right)^2}$$

$$= \frac{1}{X_M}\sqrt{\frac{1}{\tan^2\left[\cos^{-1} PF\right]} + 1}$$

$$\left|Y_E\right| = \frac{1}{X_M}\sqrt{\frac{1}{\tan^2\left[\cos^{-1} PF\right]}+1}$$

$$= \frac{1}{X_M}\sqrt{\frac{1+\tan^2\left[\cos^{-1} PF\right]}{\tan^2\left[\cos^{-1} PF\right]}}$$

$$= \frac{1}{X_M}\frac{\sec\left[\cos^{-1} PF\right]}{\tan\left[\cos^{-1} PF\right]}$$

$$= \frac{1}{X_M}\frac{1}{\cos\left[\cos^{-1} PF\right]}\frac{\cos\left[\cos^{-1} PF\right]}{\sin\left[\cos^{-1} PF\right]}$$

$$\left|Y_E\right| = \frac{1}{X_M}\frac{1}{\sin\left[\cos^{-1} PF\right]}$$

$$\Rightarrow X_M = \frac{\left|Z_E\right|}{\sin\left[\cos^{-1} PF\right]}$$

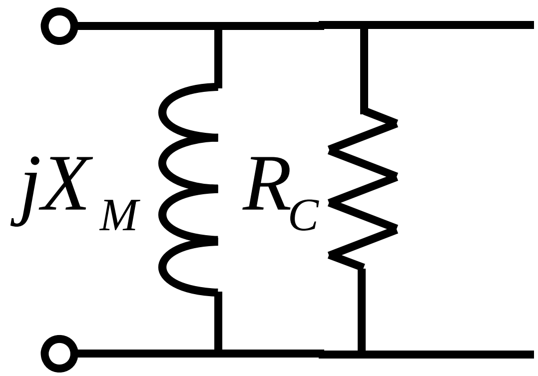
$$\Rightarrow R_C = X_M \tan\left[\cos^{-1} PF\right]$$

$$= \frac{\left|Z_E\right|}{\sin\left[\cos^{-1} PF\right]}\tan\left[\cos^{-1} PF\right] = \frac{\left|Z_E\right|}{\cos\left[\cos^{-1} PF\right]}$$

$$= \frac{\left|Z_E\right|}{PF}$$

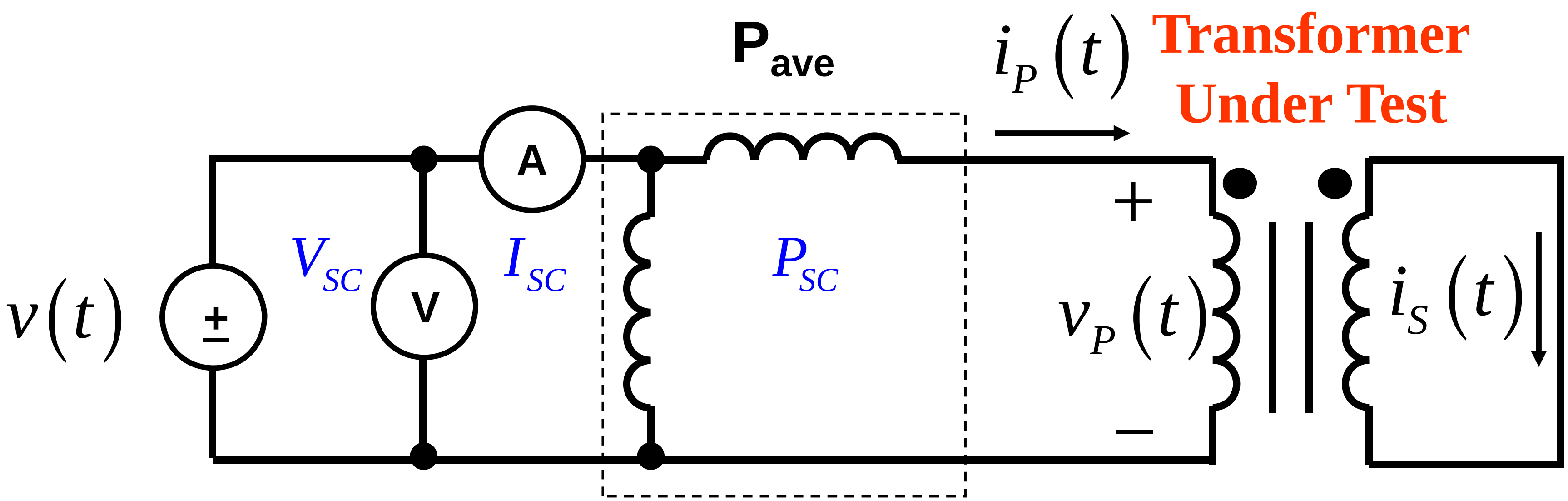
$$X_M = \frac{|Z_E|}{\sin\left[\cos^{-1} PF\right]}, \quad R_C = \frac{|Z_E|}{PF}$$

$$Z_E = \frac{1}{\frac{1}{jX_M} + \frac{1}{R_C}} \longrightarrow$$



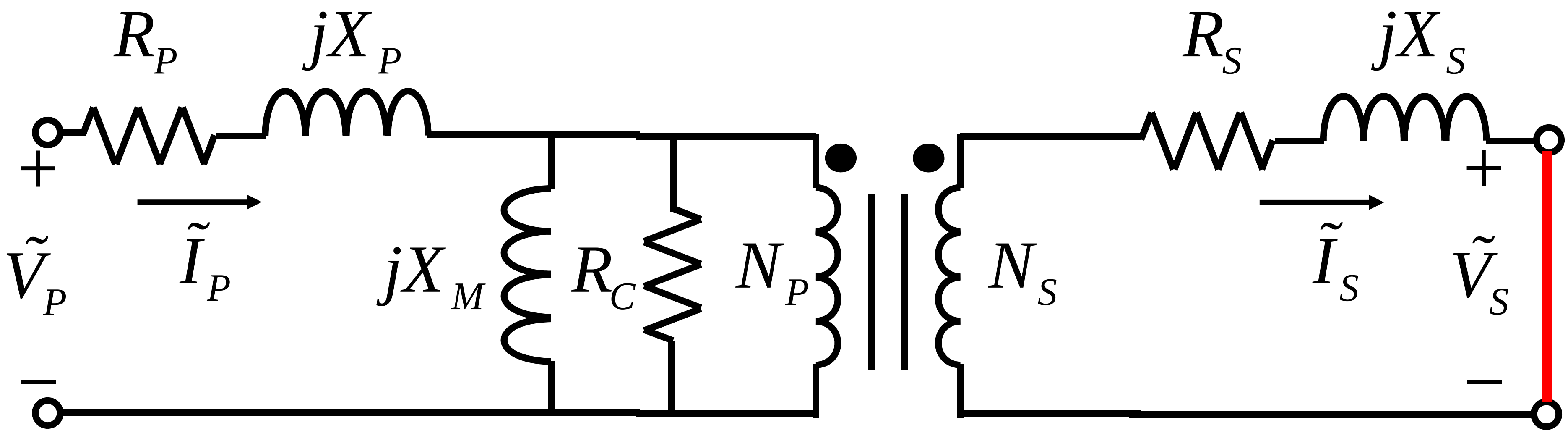
PF	R_C	$\lvert X_C \rvert$
0	$\infty\,(open)$	Z_E
1	Z_E	$\infty\,(open)$

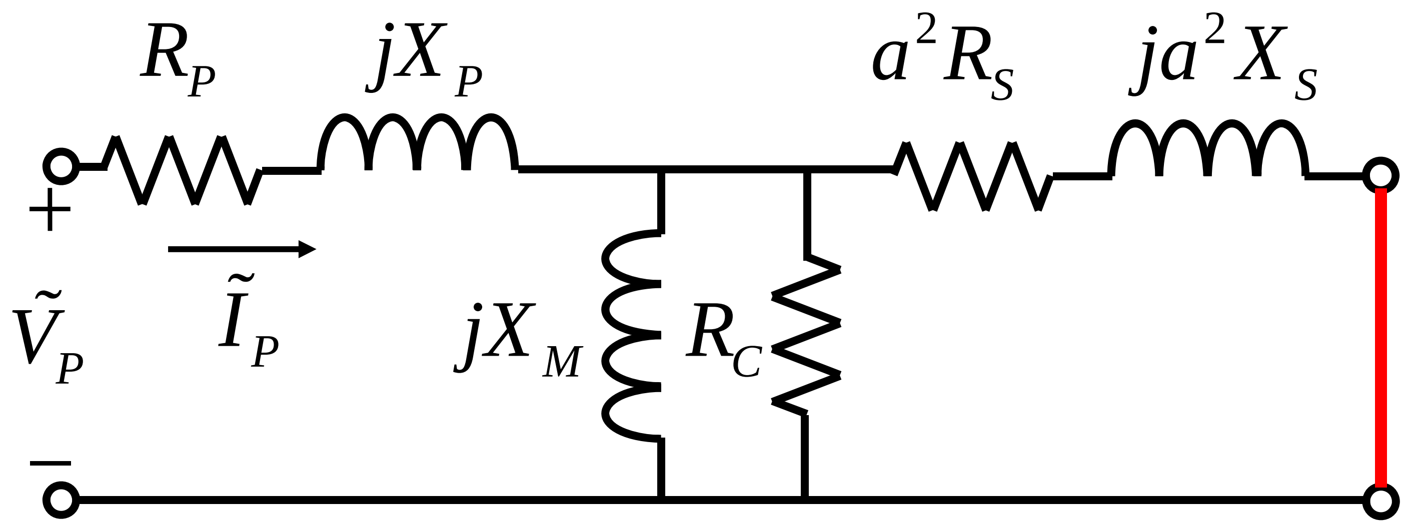
Short Circuit Test



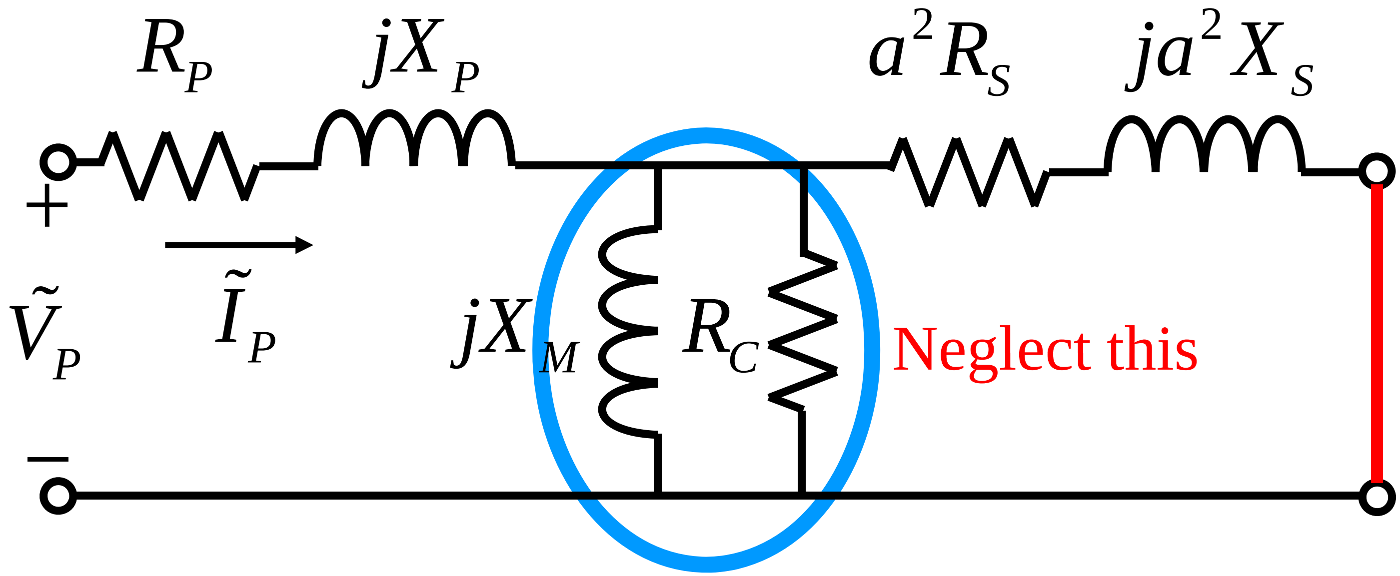
Begin with a small voltage for $v(t)$ and increase it until $i_S(t)$ is at its rated value.

Short Circuit Test



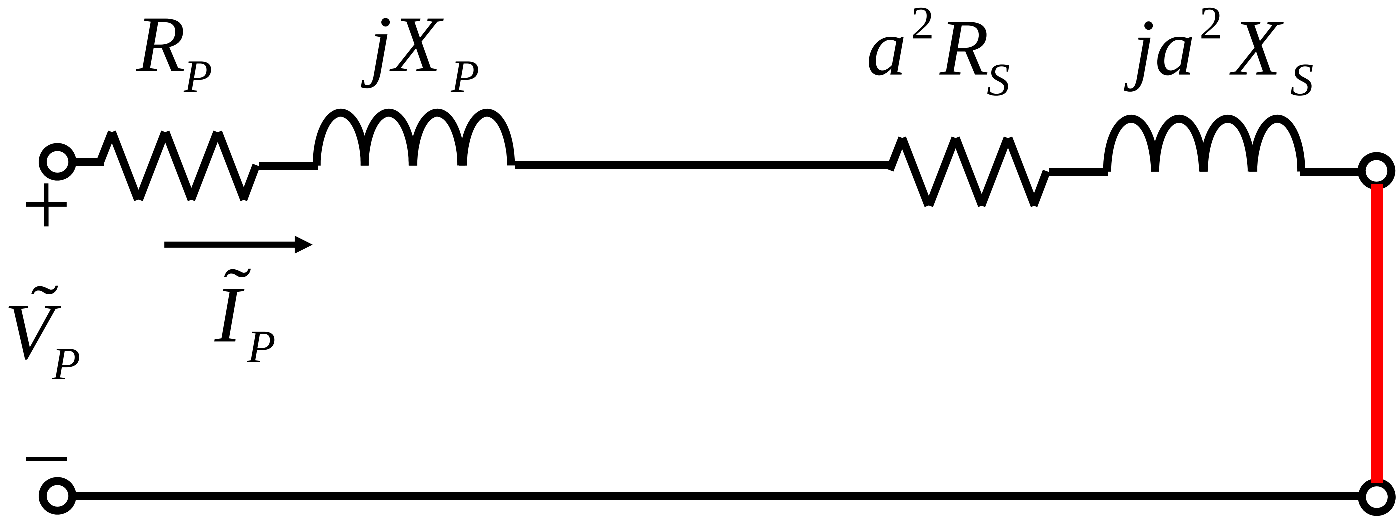


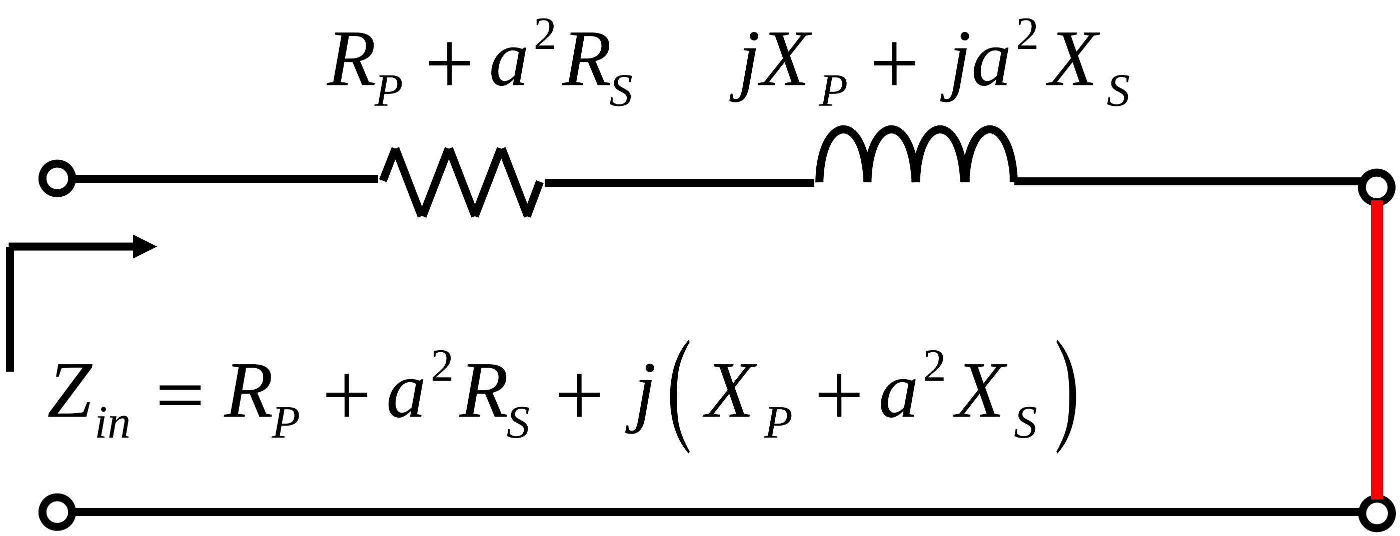
Short Circuit Test



Since the shunt magnetizing impedance is generally large compared with the series impedance $R_P + jX_P$, the shunt impedance can be neglected..

Short Circuit Test





$$Z_{in} = R_P + a^2R_S + j\left(X_P + a^2X_S\right)$$

Transformer Voltage Regulation and Efficiency

Since any real transformer has an internal impedance (resistance), the output voltage varies with the load even if the input voltage remains constant.

To quantify this effect, it is customary to define the voltage regulation (VR) of the transformer as:

$$VR = \frac{V_{Secondary,\, no\ load} - V_{Secondary,\, full\ load}}{V_{Secondary,\, full\ load}} \times 100\%$$

Transformer Voltage Regulation and Efficiency

$$VR = \frac{V_{Secondary,\, no\ load} - V_{Secondary,\, full\ load}}{V_{Secondary,\, full\ load}} \times 100\%$$

$$V_{Secondary,\, no\ load} = \frac{V_{Primary}}{a}$$

$$\Rightarrow VR = \frac{\dfrac{V_{Primary}}{a} - V_{Secondary,\, full\ load}}{V_{Secondary,\, full\ load}} \times 100\%$$

Transformer Voltage Regulation and Efficiency

$$\eta = \frac{P_{out}}{P_{in}} \times 100\%$$

$$= \frac{P_{out}}{P_{out} + P_{loss}} \times 100\%$$

Losses:

Copper (I^2R)
Hysteresis (R_C)
Eddy Current

Transformer Voltage Regulation and Efficiency

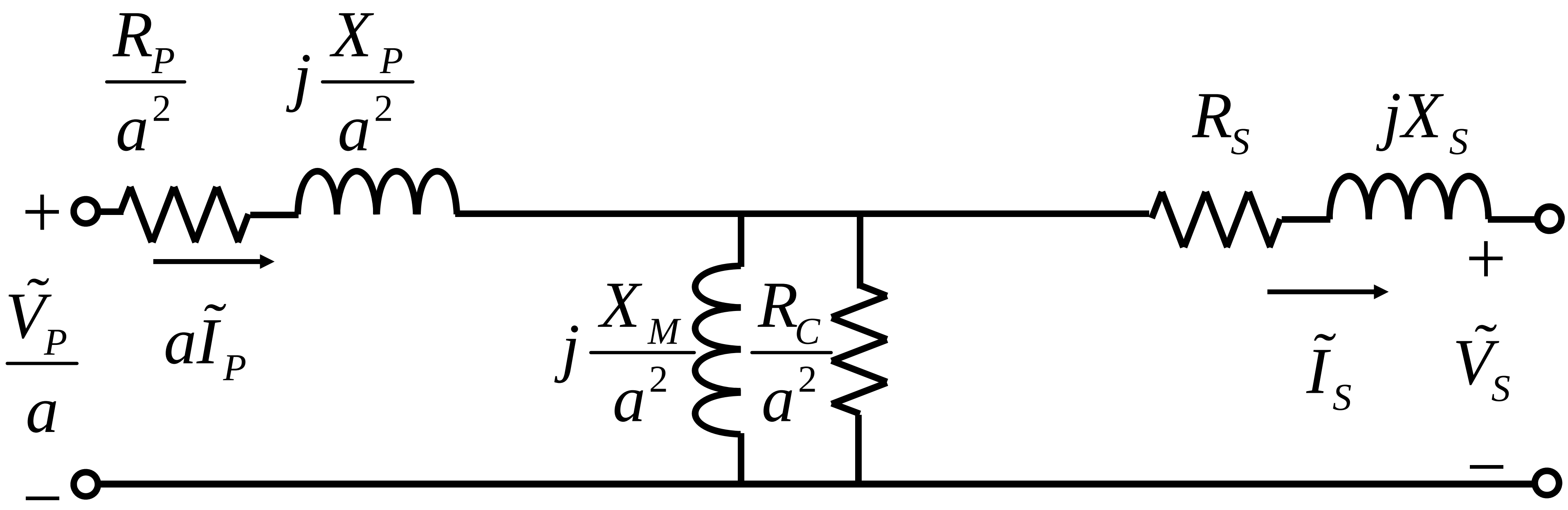
$$\eta = \frac{P_{out}}{P_{out} + P_{loss}} \times 100\%$$

$$= \frac{V_S I_S \cos\theta}{V_S I_S \cos\theta + P_{Copper} + \underbrace{P_{Hysteresis} + P_{Eddy}}_{core\ losses}} \times 100\%$$

$$= \frac{V_S I_S \cos\theta}{V_S I_S \cos\theta + P_{Copper} + P_{Core}} \times 100\%$$

Transformer Phasor Diagrams – A useful visualization tool

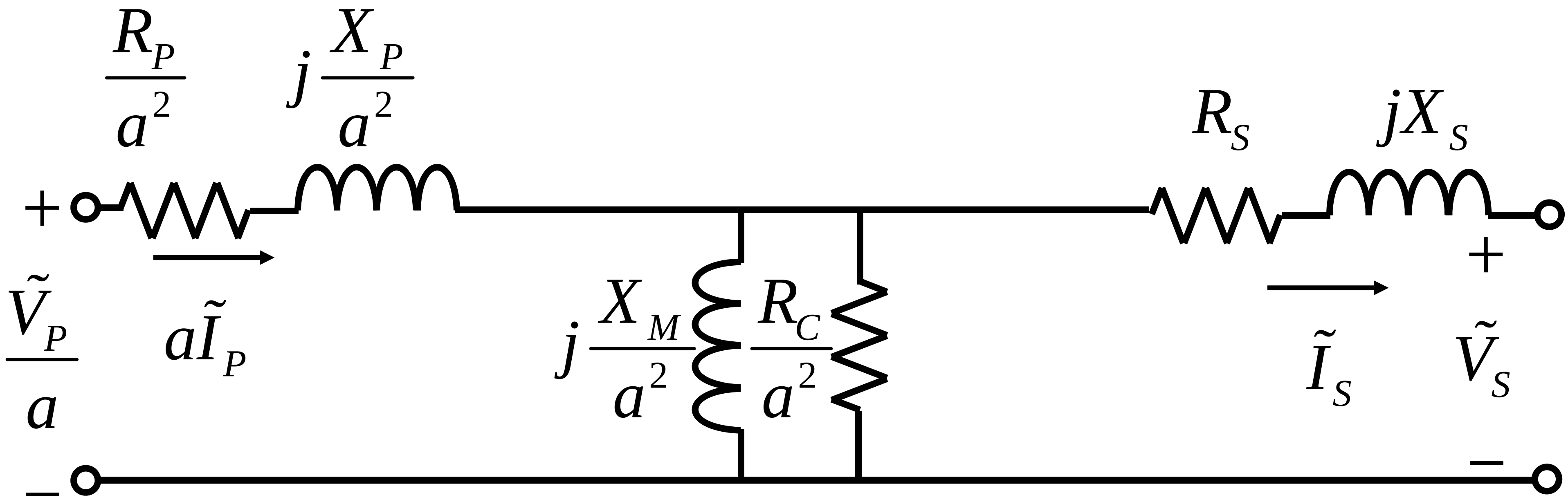
Recall our earlier model:



Model referred to its secondary side

Transformer Phasor Diagrams – A useful visualization tool

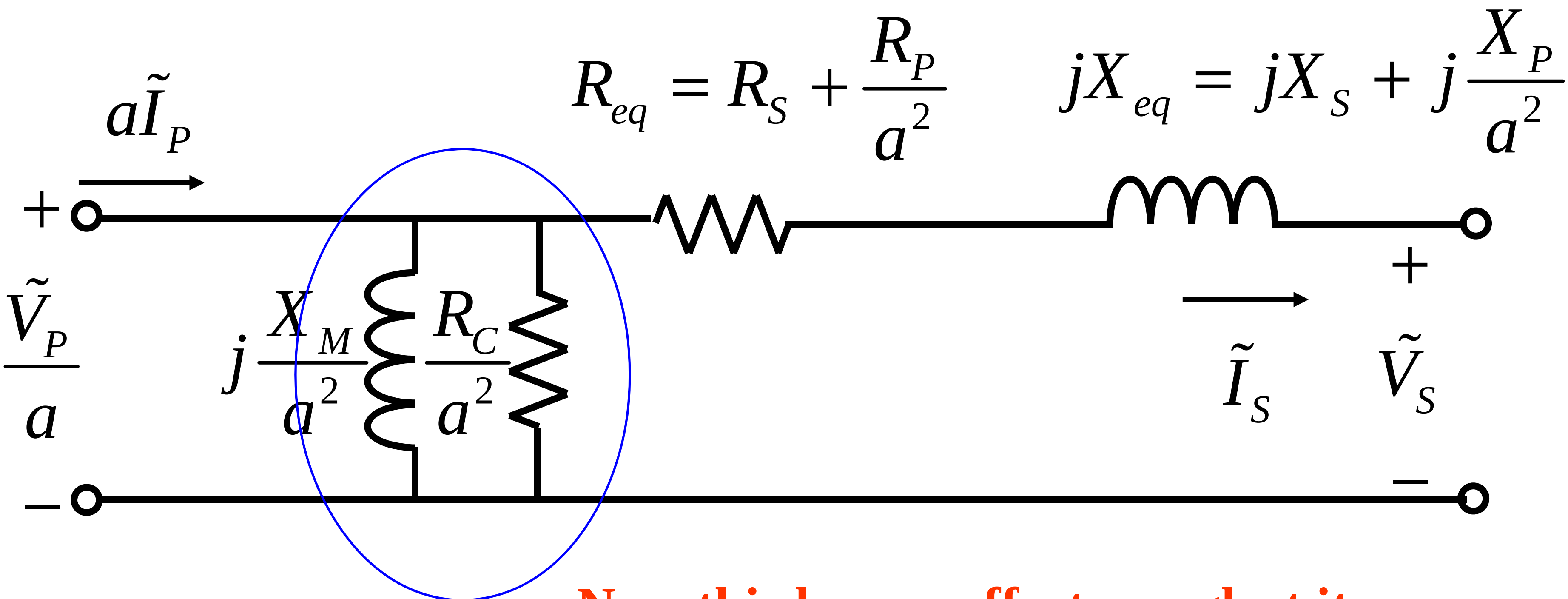
Recall our earlier model:



The excitation branch will have little effect on voltage regulation.

Transformer Phasor Diagrams – A useful visualization tool

Recall our earlier model:



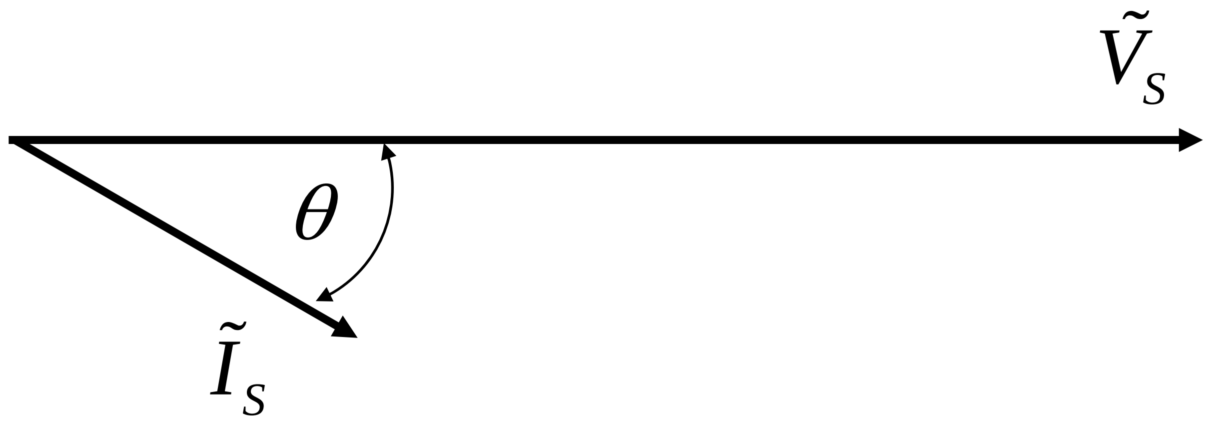
Now this has no effect – neglect it

$$\frac{\tilde{V}_P}{a} = \tilde{I}_S R_{eq} + j\tilde{I}_S X_{eq} + \tilde{V}_S$$

Transformer Phasor Diagrams – A useful visualization tool for this equation:

$$\frac{\tilde{V}_P}{a} = \tilde{I}_S R_{eq} + j\tilde{I}_S X_{eq} + \tilde{V}_S$$

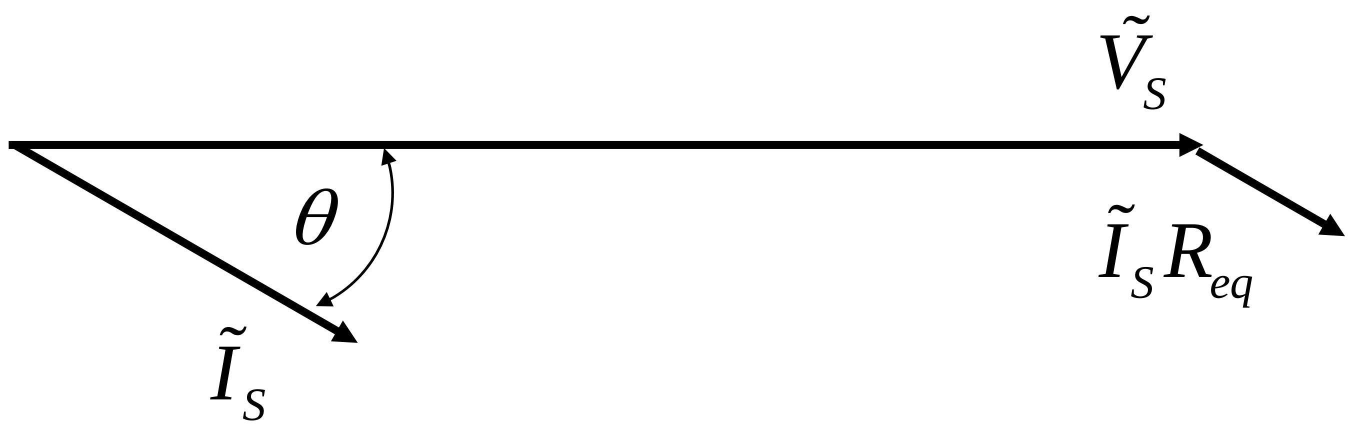
Case 1: Lagging Power Factor



Transformer Phasor Diagrams – A useful visualization tool for this equation:

$$\frac{\tilde{V}_P}{a} = \tilde{I}_S R_{eq} + j\tilde{I}_S X_{eq} + \tilde{V}_S$$

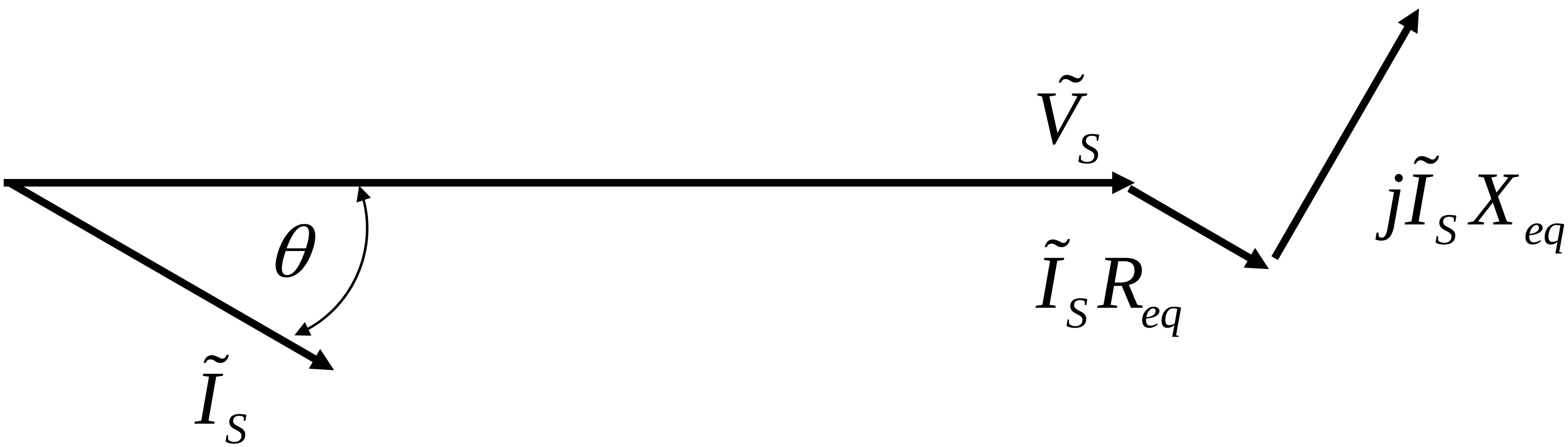
Case 1: Lagging Power Factor



Transformer Phasor Diagrams – A useful visualization tool for this equation:

$$\frac{\tilde{V}_P}{a} = \tilde{I}_S R_{eq} + j\tilde{I}_S X_{eq} + \tilde{V}_S$$

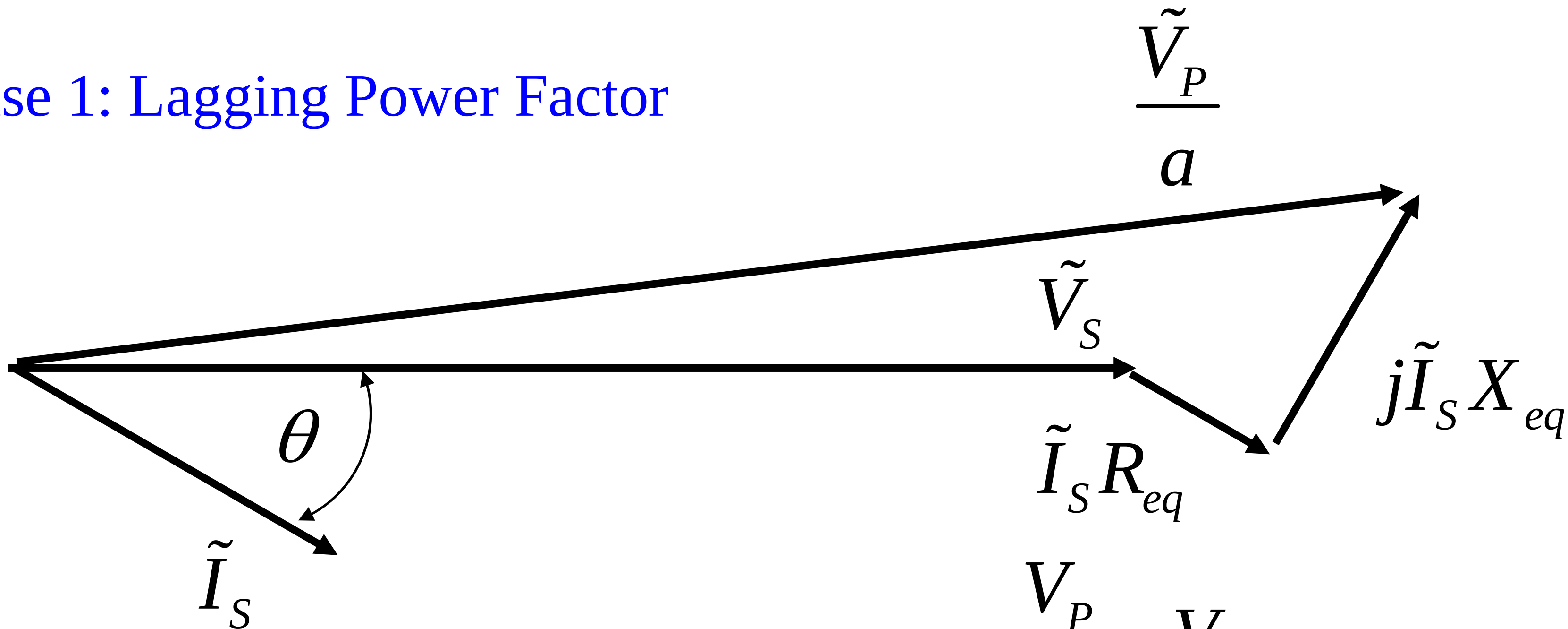
Case 1: Lagging Power Factor



Transformer Phasor Diagrams – A useful visualization tool for this equation:

$$\frac{\tilde{V}_P}{a} = \tilde{I}_S R_{eq} + j\tilde{I}_S X_{eq} + \tilde{V}_S$$

Case 1: Lagging Power Factor

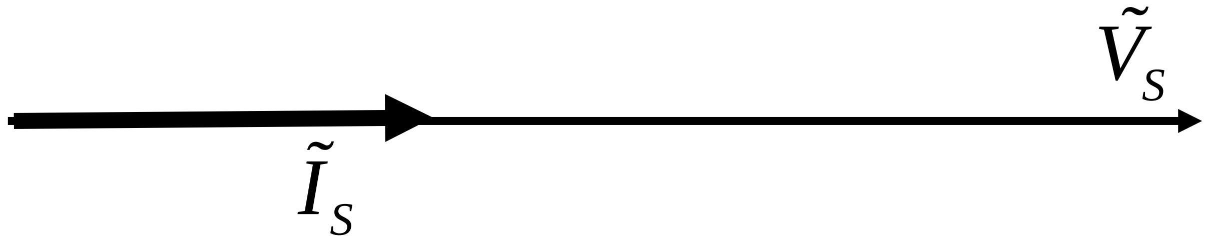


For lagging loads: $\frac{V_P}{a} > V_S \Rightarrow VR = \frac{\frac{V_P}{a} - V_S}{V_S} > 0$

Transformer Phasor Diagrams – A useful visualization tool for this equation:

$$\frac{\tilde{V}_P}{a} = \tilde{I}_S R_{eq} + j\tilde{I}_S X_{eq} + \tilde{V}_S$$

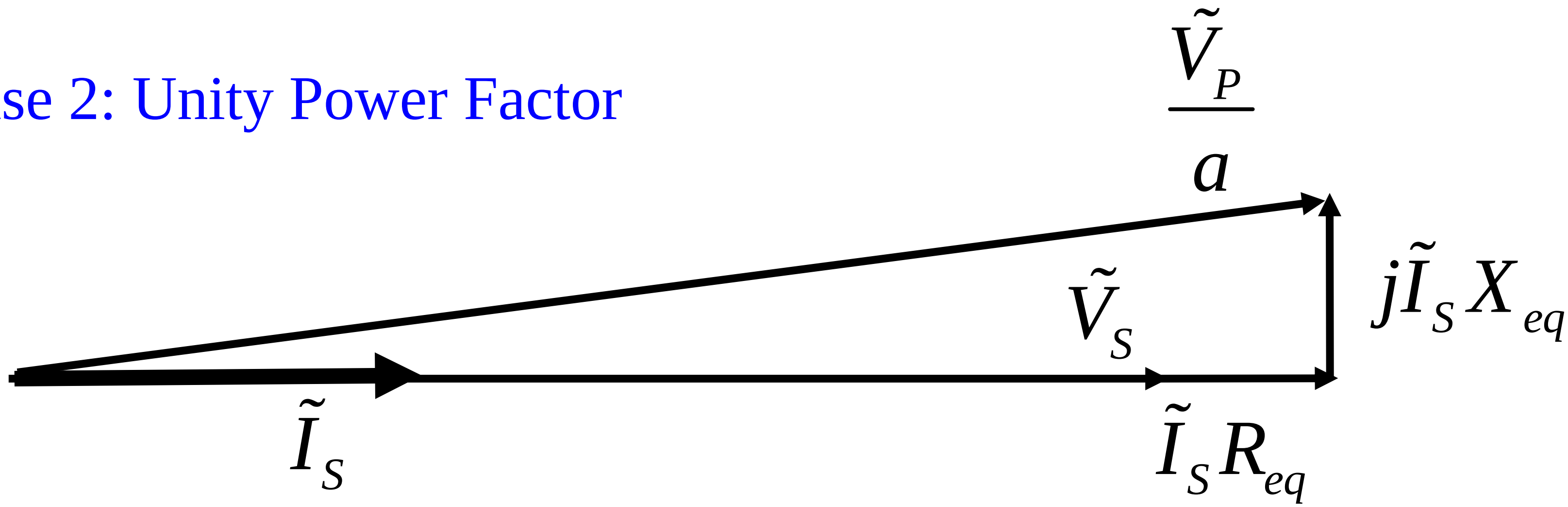
Case 2: Unity Power Factor



Transformer Phasor Diagrams – A useful visualization tool for this equation:

$$\frac{\tilde{V}_P}{a} = \tilde{I}_S R_{eq} + j\tilde{I}_S X_{eq} + \tilde{V}_S$$

Case 2: Unity Power Factor

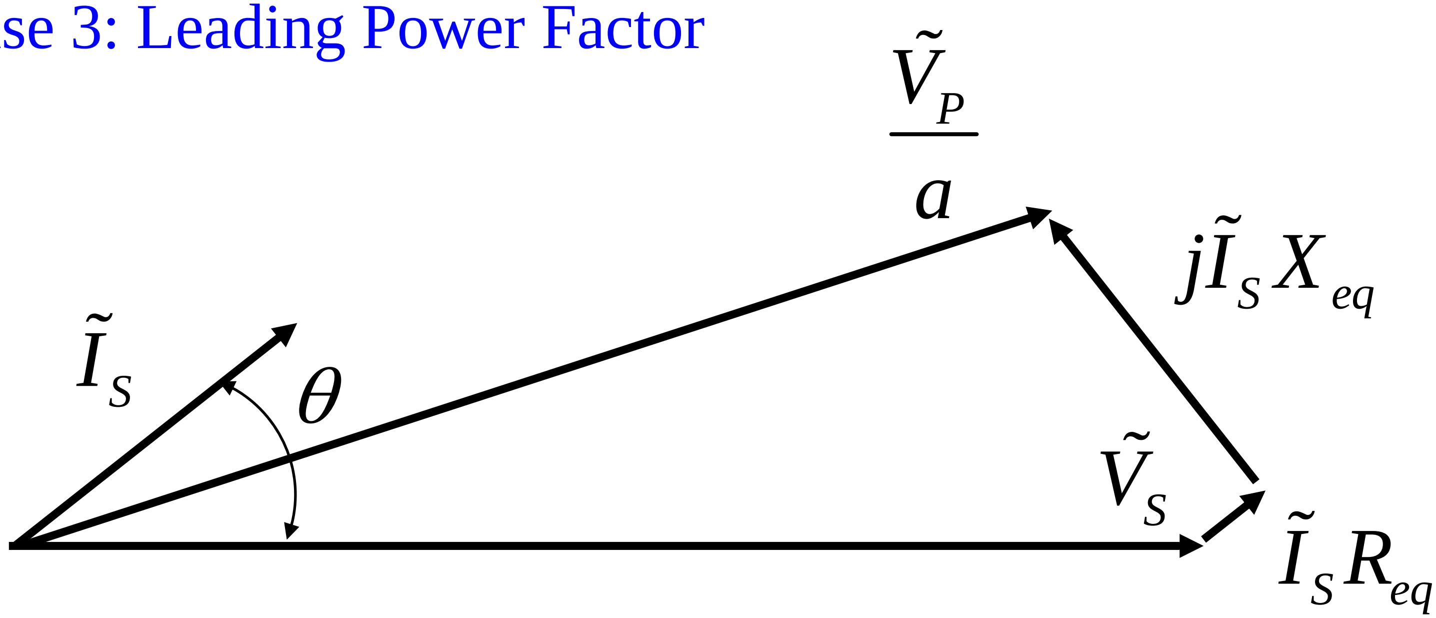


For unity PF: $\frac{V_P}{a} > V_S \Rightarrow VR = \frac{\frac{V_P}{a} - V_S}{V_S} > 0$

Note that *VR* is smaller here than in the previous case.

Transformer Phasor Diagrams – A useful visualization tool for this equation:

Case 3: Leading Power Factor



For leading loads: $\frac{V_P}{a} < V_S \Rightarrow VR = \frac{\frac{V_P}{a} - V_S}{V_S} < 0$ Note that *VR* is could be negative.

Example

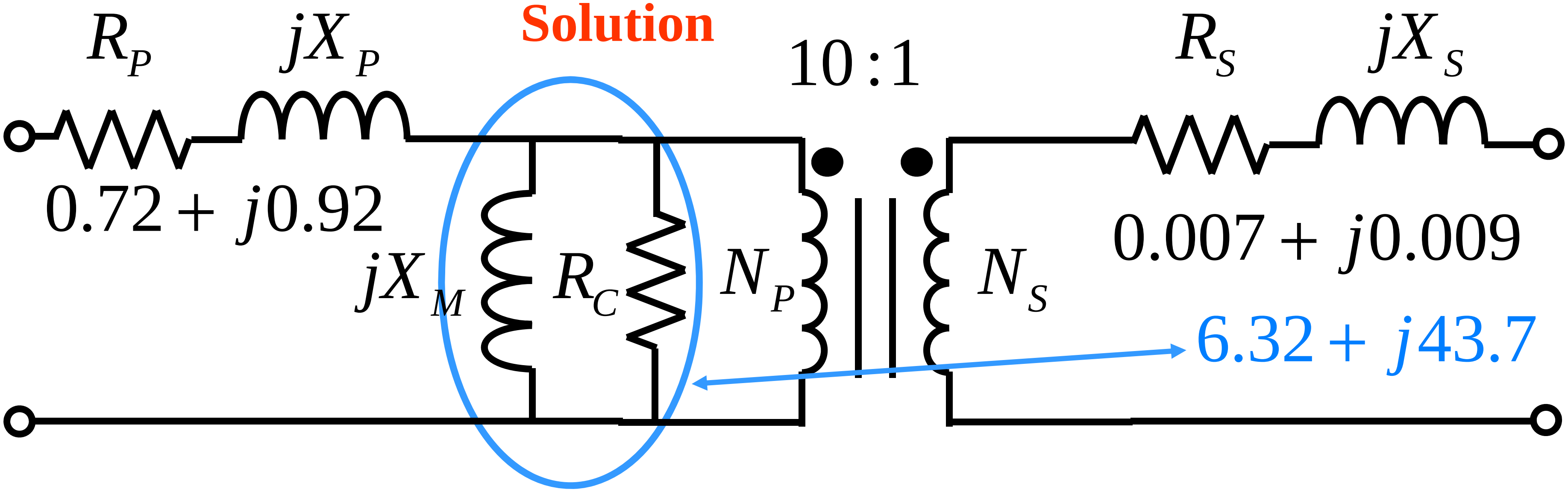
$V_P : V_S$

A 50 kVA, 2400:240 V, 60 Hz distribution transformer has a leakage impedance of 0.72 + *j* 0.92 Ohms in the high-voltage winding and 0.0070 + *j* 0.0090 Ohms in the low-voltage winding. At the rated frequency, the impedance of the shunt exciting branch (the impedance of magnetization branch R_C and jX_M in parallel) is 6.32 + *j* 43.7 Ohms *when viewed from the low-voltage side*. Draw the equivalent circuit referred to

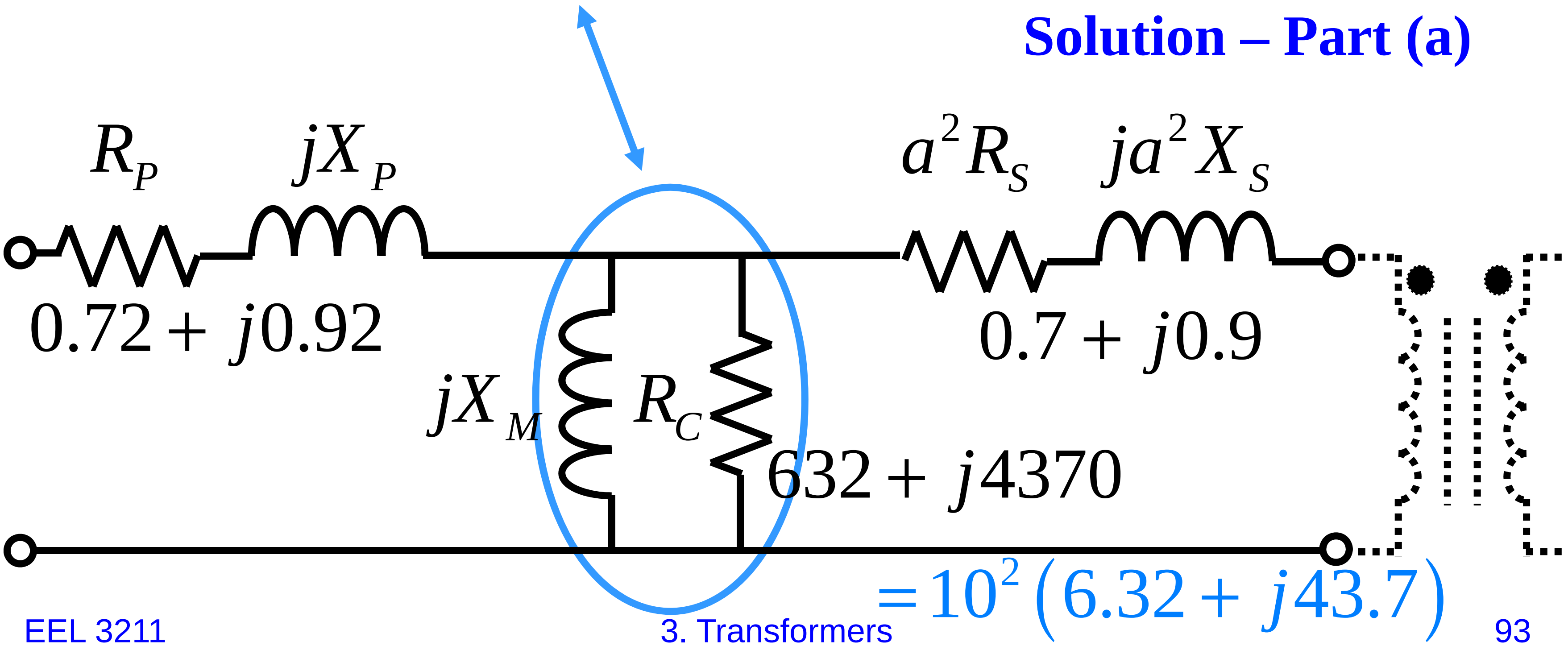
a) the high-voltage side and

b) the low-voltage side.

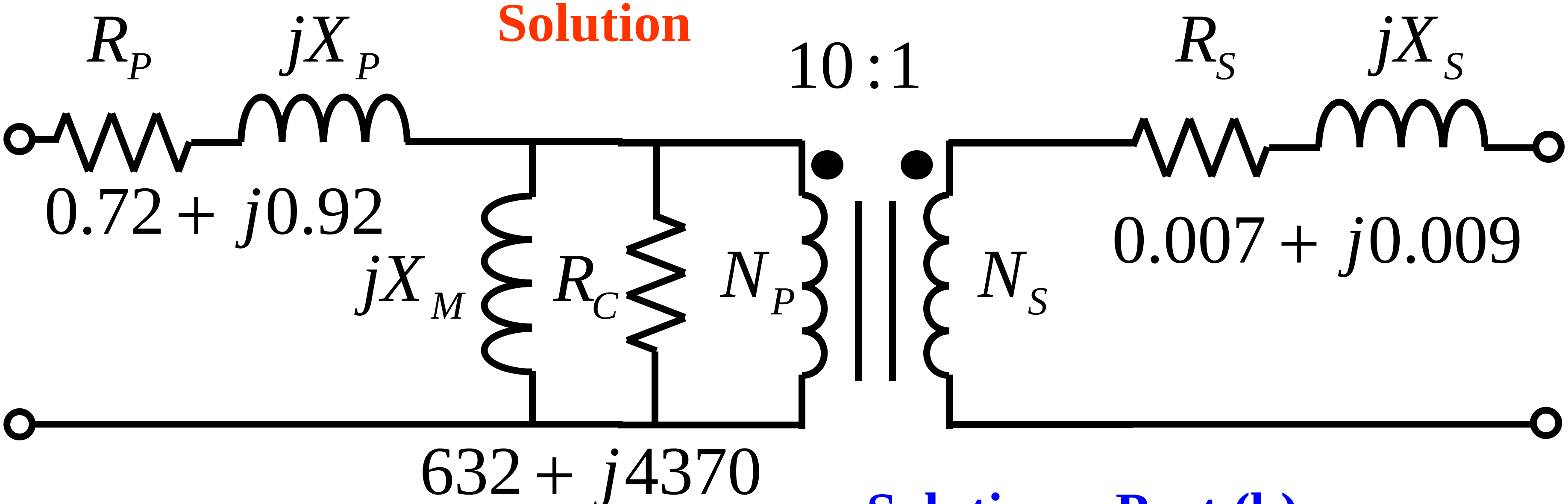
Solution



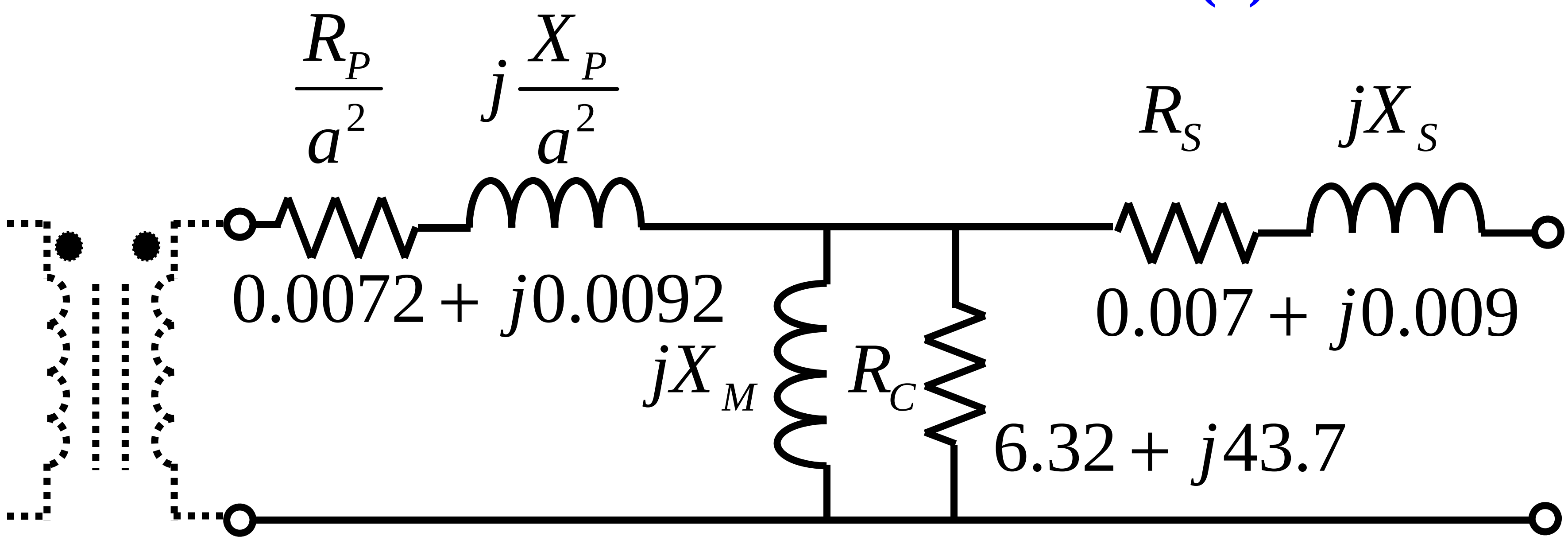
Solution – Part (a)



Solution



Solution – Part (b)



Example – Continued **Solution**

If 2400 V *rms* is applied to the high voltage side of the transformer, calculate the current in the magnetizing impedance in parts (a) and (b), respectively.

Solution (a):

$$Z_T = 632 + 0.72 + j4370 + j0.92$$

$$= 632.72 + j4370.92$$

$$= 4416.5\angle 81.76°$$

$$\Rightarrow I_M = \frac{2400}{4416.5\angle 81.76°} = 0.543\angle -81.76° \text{ } amps\ rms$$

Example – Continued Solution

If 2400 V *rms* is applied to the high voltage side of the transformer, calculate the current in the magnetizing impedance in parts (a) and (b), respectively.

Solution (b):

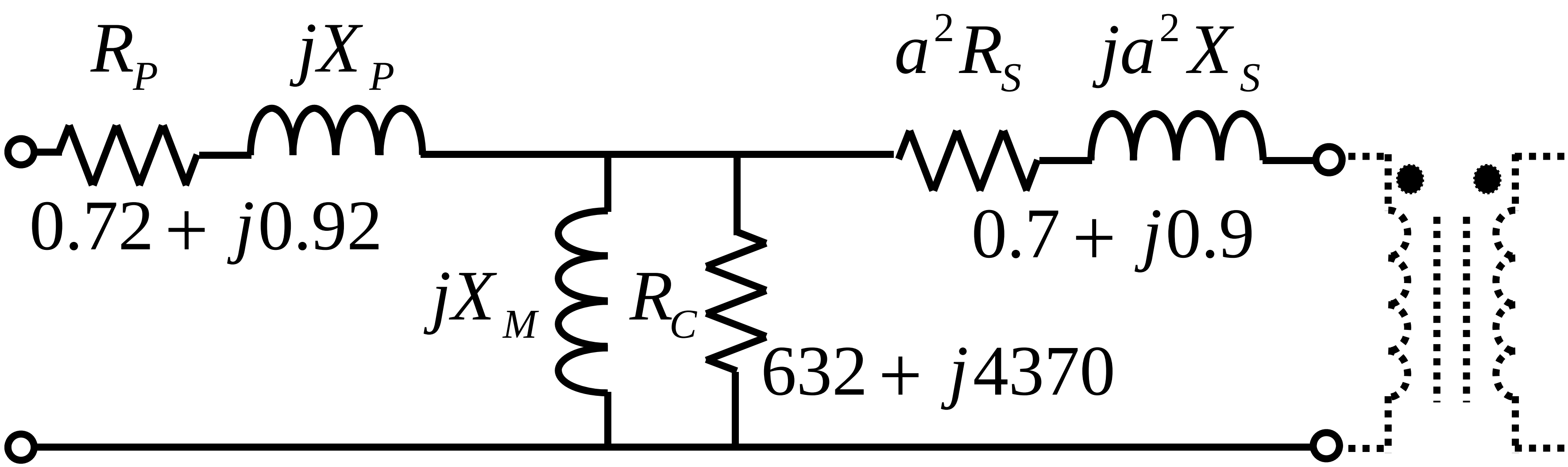
$$Z_T = 6.3272 + j43.7092$$
$$= 44.171\angle 81.76^\circ$$
$$\Rightarrow I_M = \frac{240}{44.171\angle 81.76^\circ} = 5.43\angle -81.76^\circ \; amps \; rms$$

Example – Continued

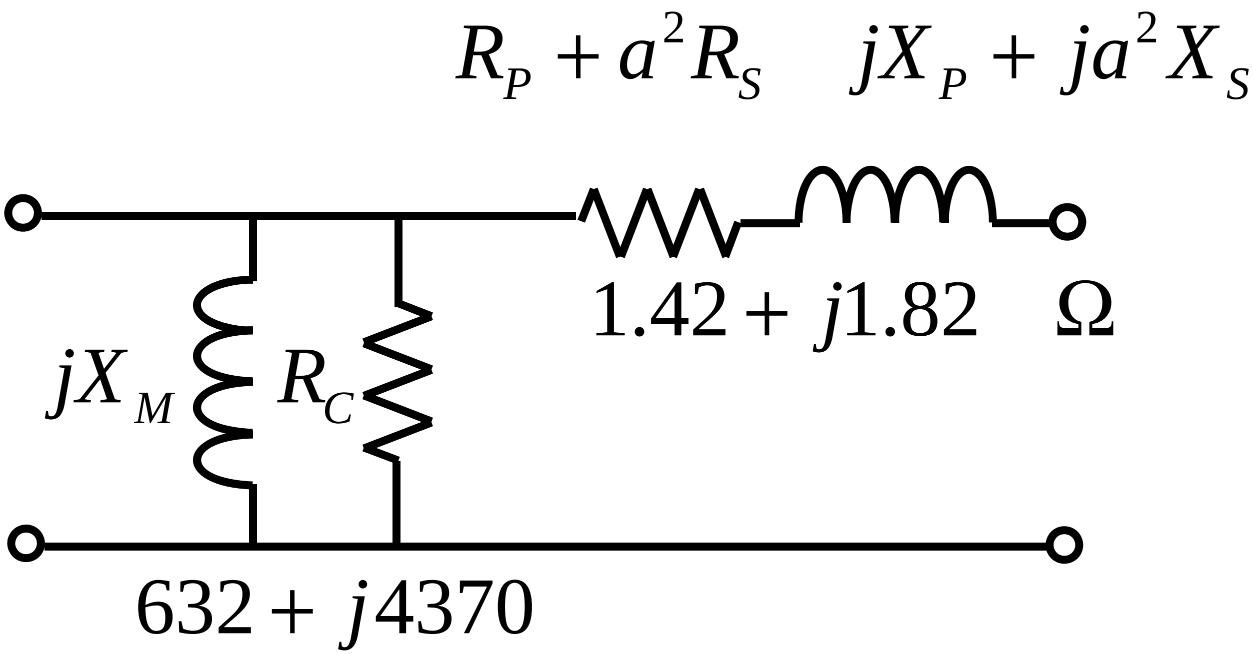
Consider the equivalent "T" circuit found with the impedances referred to the high-voltage side (below).

(a)Draw the "cantilever" equivalent circuit with the shunt branch at the high-voltage terminal.

(b) With the low-voltage terminal an open-circuited and 2400 V applied to the high-voltage terminal, calculate the voltage at the low-voltage terminal as predicted by each equivalent circuit.



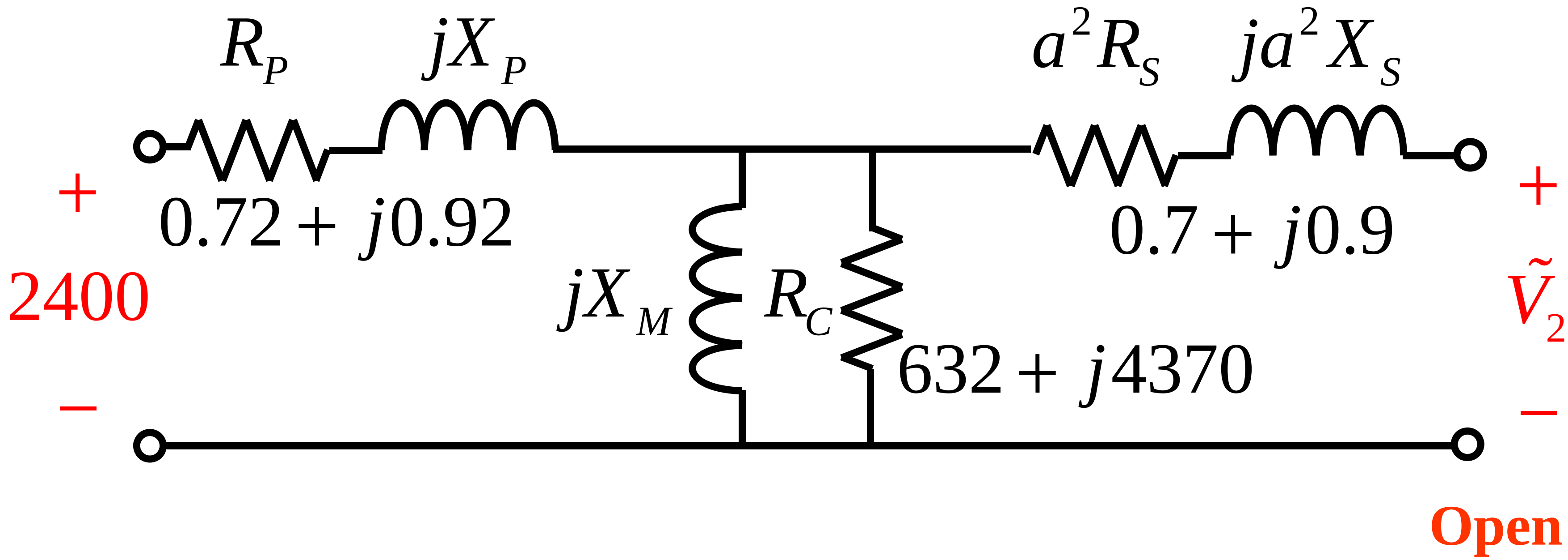
(a) Draw the "cantilever" equivalent circuit with the shunt branch at the high-voltage terminal.



Example – Continued

Solution

(b) With the low-voltage terminal an open-circuit and 2400 V applied to the high-voltage terminal, calculate the voltage at the low-voltage terminal as predicted by each equivalent circuit.



Example – Continued (b) For the "T": **Solution**

$$\tilde{V}_2 = \frac{Z_M}{Z_M + Z_P} 2400$$

$$= \frac{632 + j4370}{632 + j4370 + 0.72 + j0.92} 2400$$

$$\frac{632 + j \cdot 4370}{632.72 + j \cdot 4370.92} \cdot 2400 = 2.399 \times 10^3 + 0.316i$$

$$\left| \frac{632 + j \cdot 4370}{632.72 + j \cdot 4370.92} \cdot 2400 \right| = 2.3994 \times 10^3$$

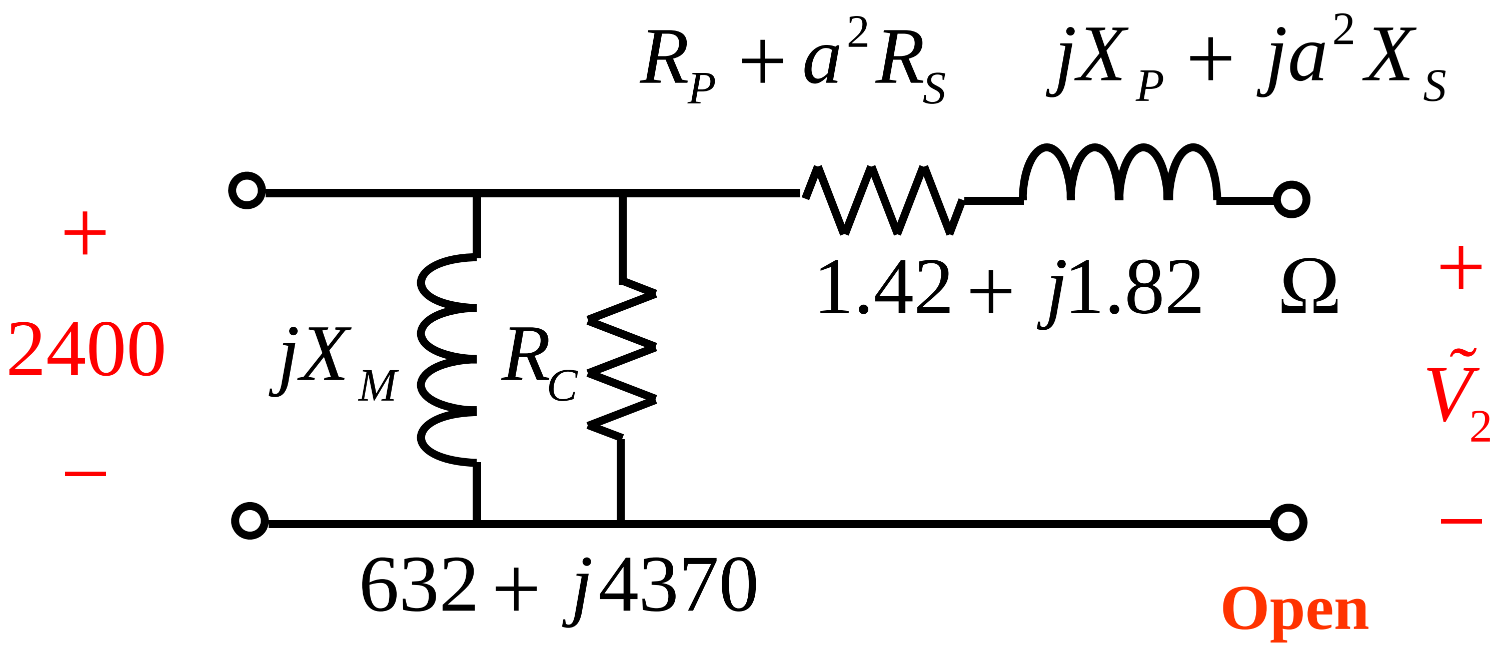
$$\frac{180}{\pi} \cdot \arg\left(\frac{632 + j \cdot 4370}{632.72 + j \cdot 4370.92} \cdot 2400 \right) = 7.536 \times 10^{-3}$$

Example – Continued (b) For the "T": **Solution**

Reflected back to the secondary side gives,

$$\left|\tilde{V}_S\right| = \frac{\left|\tilde{V}_2\right|}{10} = 239.94\,V$$

Example – Continued (b) For the "Cantilever": **Solution**



$$\tilde{V}_2 = 2400\,V \Rightarrow \tilde{V}_S = 240$$

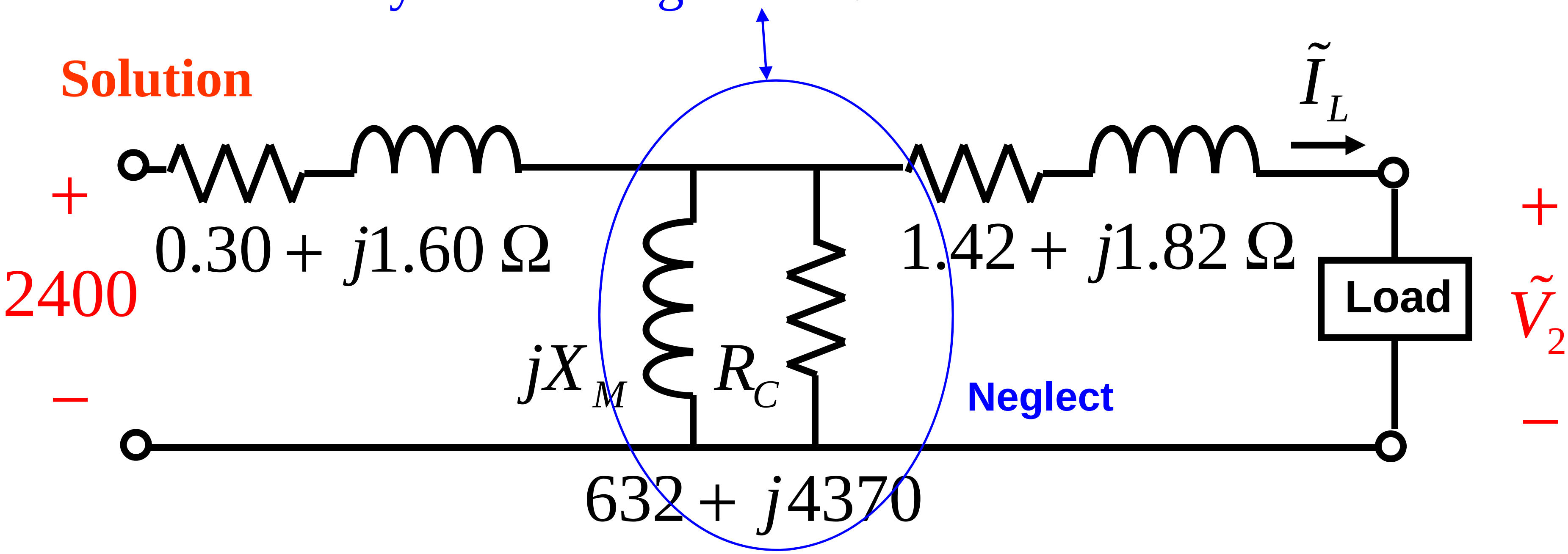
$$error = \frac{240 - 239.94}{240} \times 100\% = 0.025\%$$

Well within engineering accuracy

Example – Continued

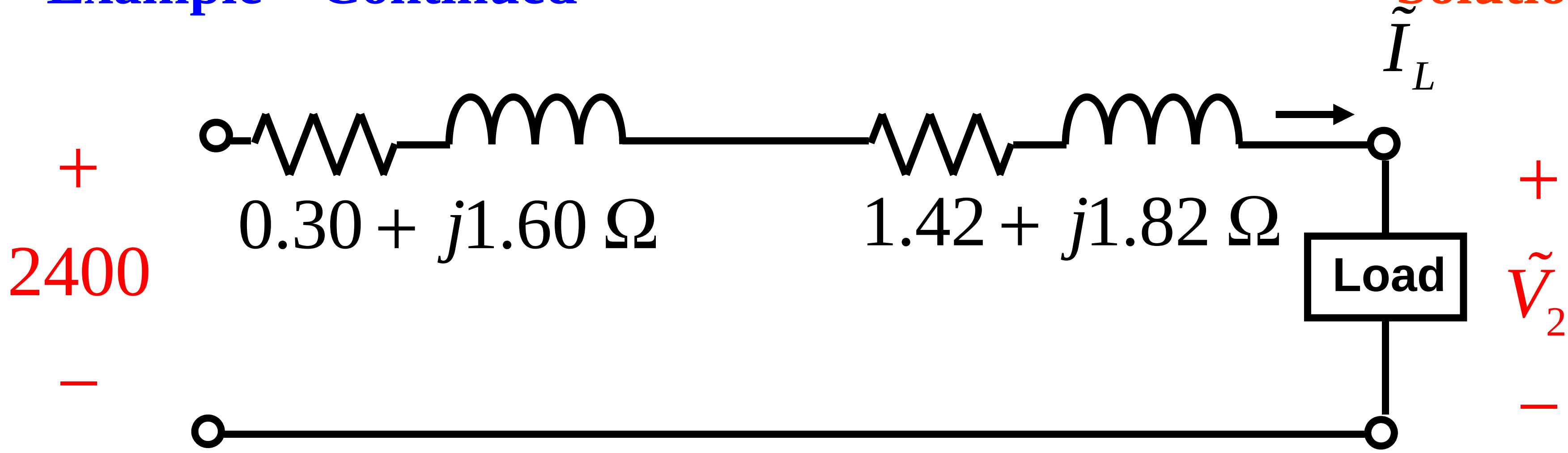
Suppose that the 2400 V feeder has an internal impedance of 0.30 + j1.60 Ohms. Find the voltage at the secondary terminals of the transformer when the load connected to the secondary draws rated current from the transformer and the power factor of the load is 0.80 lagging. Neglect the voltage drops in the transformer and the feeder caused by the exciting current.

Solution



Example – Continued

Solution



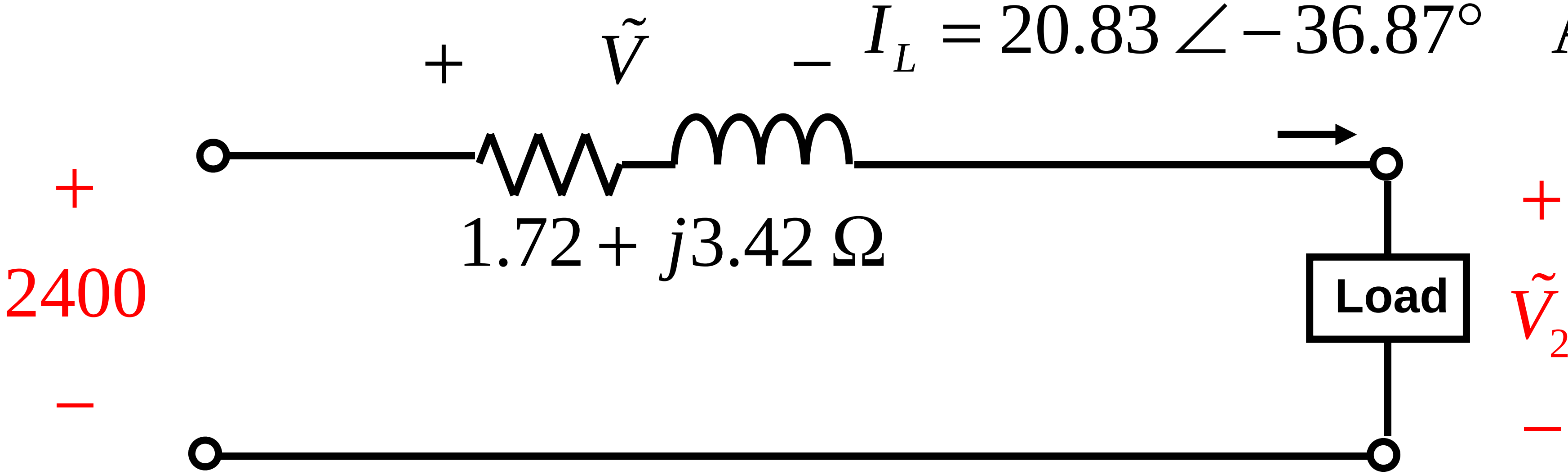
This is the high-voltage end:

$$\left|\tilde{I}_L\right| = \frac{50\text{-kVA}}{2400} = 20.83\text{ A}$$

$$\theta = \underset{\substack{\uparrow \\ \textit{lagging}}}{-} \cos^{-1} 0.8 = -36.87^\circ$$

Example – Continued Solution

$$\tilde{I}_L = 20.83\angle -36.87^\circ \quad \text{A}$$



$$\tilde{V} = Z\tilde{I}_L = (1.72 + j3.42)\,20.83\angle -36.87^\circ \quad V$$

$$\tilde{V}_2 = 2400 - \tilde{V} = 2400 - \left[(1.72 + j3.42)\,20.83\angle -36.87^\circ\right]$$

$$\left|2400 - (1.72 + j\cdot 3.42)\cdot 20.83\cdot e^{-j\cdot 36.87\cdot\frac{\pi}{180}}\right| = 2.329\times 10^{3}$$

Example – Continued

Solution

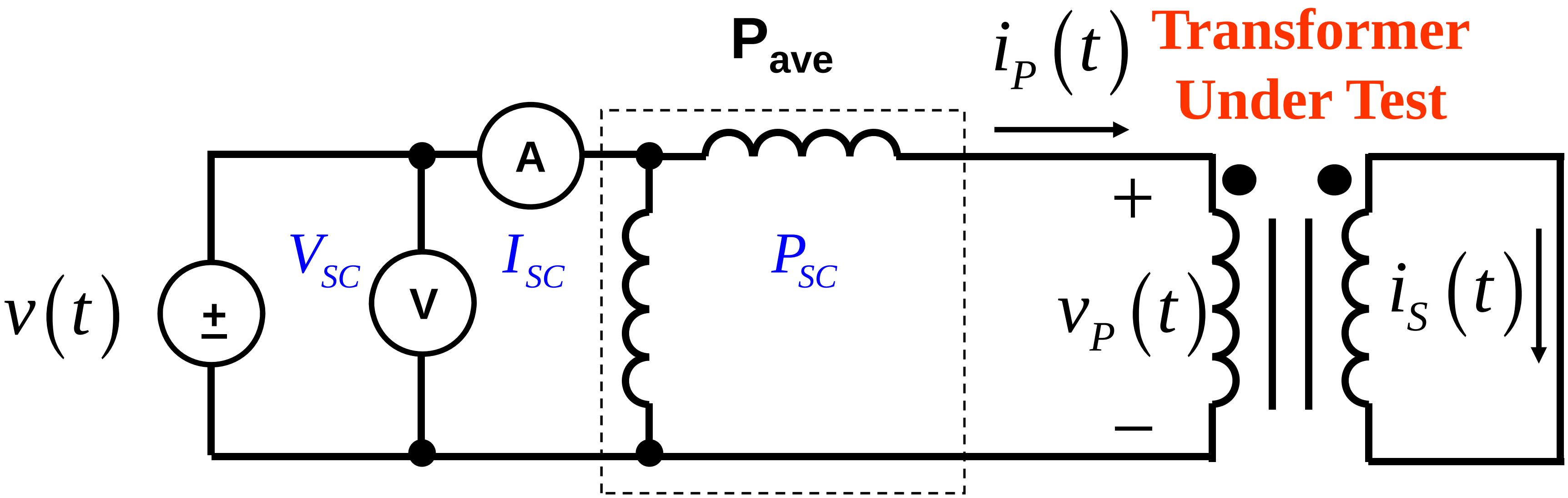
$$\left|\tilde{V}_2\right| = 2329$$

At the load,

$$\left|\tilde{V}_L\right| = \frac{\left|\tilde{V}_2\right|}{10} = 232.9$$

Example – Continued

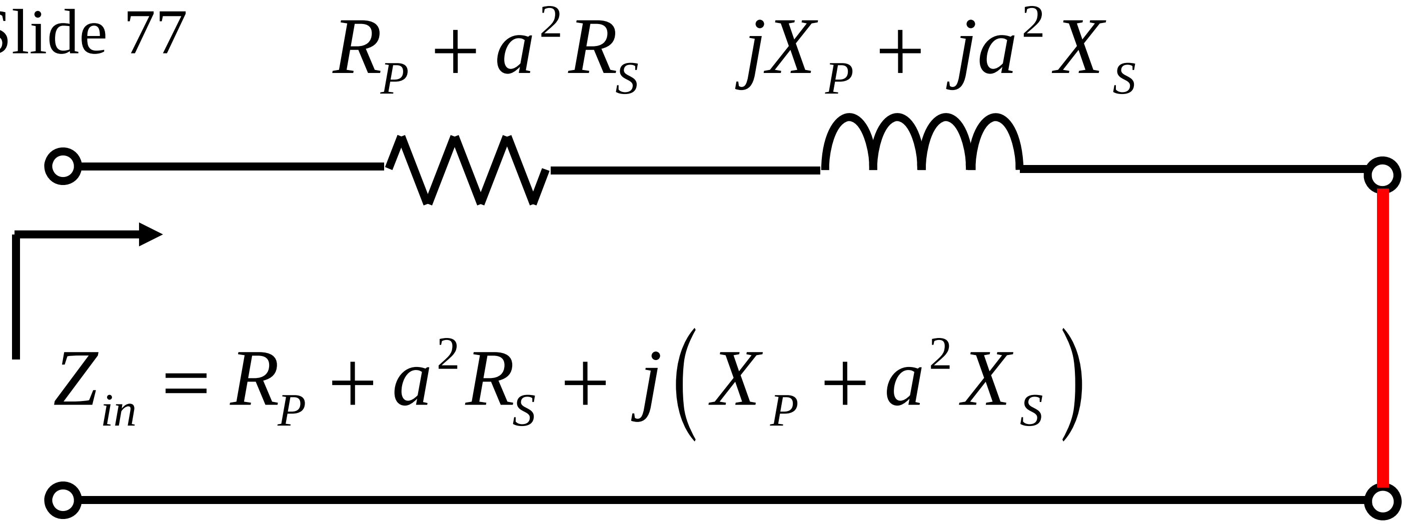
With instrumentation on the high-voltage side and the output short-circuited, the short-circuit test readings for this example are 48 V, 20.8 A, and 617 W. An open circuit test with the low-voltage side energized gives instrument readings on that side of 240 V, 5.41 A, and 186 W. Determine the efficiency and the voltage regulation at full load, 0.80 power factor lagging.



Example – Continued

Solution

From Slide 77



$$|Z_{in}| = \frac{|V|}{|I|} = \frac{48}{20.8} = 2.31\,\Omega$$

$$P = |I|^2 R_{eq} \Rightarrow R_{eq} = \frac{617}{20.8^2} = 1.42\,\Omega$$

Example – Continued **Solution**

$$|Z_{in}|^2 = \left(R_P + a^2 R_S\right)^2 + \left(X_P + a^2 X_S\right)^2$$

$$\Rightarrow \left(X_P + a^2 X_S\right)^2 = |Z_{in}|^2 - \left(R_P + a^2 R_S\right)^2$$

$$X_{eq} = \sqrt{|Z_{in}|^2 - R_{eq}{}^2}$$

$$= \sqrt{2.31^2 - 1.42^2}$$

$$= 1.82\ \Omega$$

Example – Continued **Solution**

Operation at full load, 0.80 power factor, lagging, corresponds to a primary current of

$$\left|\tilde{I}_P\right| = \frac{VI}{\left|V_P\right|} = \frac{50,000\text{-kVA}}{2400} = 20.8 \text{ A}$$

and an output power of

$$P_{av} = VI\cos\theta = 50,000 \text{ VA}\bullet\left(0.8\right) = 40,000\, W$$

Example – Continued **Solution**

The loss in the windings is

$$P_{windings} = \left|\tilde{I}_P\right|^2 R_{eq} = 20.8^2 \times 1.42 = 617 \text{ W}$$

The core loss is determined from the open-circuit test,

$$P_{core} = 186 \text{ W } \left(\text{given}\right)$$

The total loss is

$$P_{winding} + P_{core} = 617 + 186 = 803 \text{ W}$$

The power supplied is thus

$$P_{ave} + P_{loss} = 40,000 + 803 = 40,803 \text{ W}$$

Example – Continued **Solution**

The efficiency is

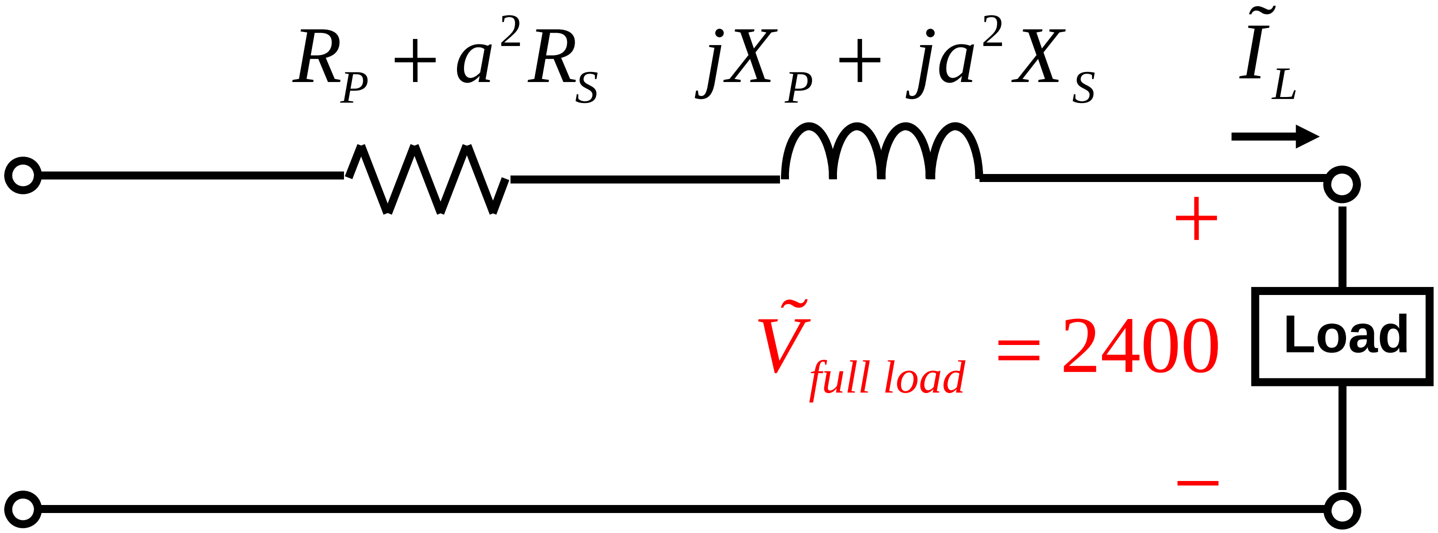
$$\eta = \frac{P_{out}}{P_{in}} \times 100\%$$

$$= \frac{P_{out}}{P_{out} + P_{loss}} \times 100\%$$

$$= \frac{40,000}{40,803} \times 100\% = 98.03\%$$

Example – Continued Voltage Regulation: **Solution**

$$VR = \frac{V_{Secondary,\,no\,load} - V_{Secondary,\,full\,load}}{V_{Secondary,\,full\,load}} \times 100\%$$



$PF = 0.8\ lagging$

Example – Continued **Solution**

Voltage Regulation:

$$\tilde{I}_L = \frac{VI}{V} e^{-j\theta} = \frac{50,000}{2,400} e^{-j36.9^\circ}$$

$$= 20.83(0.8 - j0.6)$$

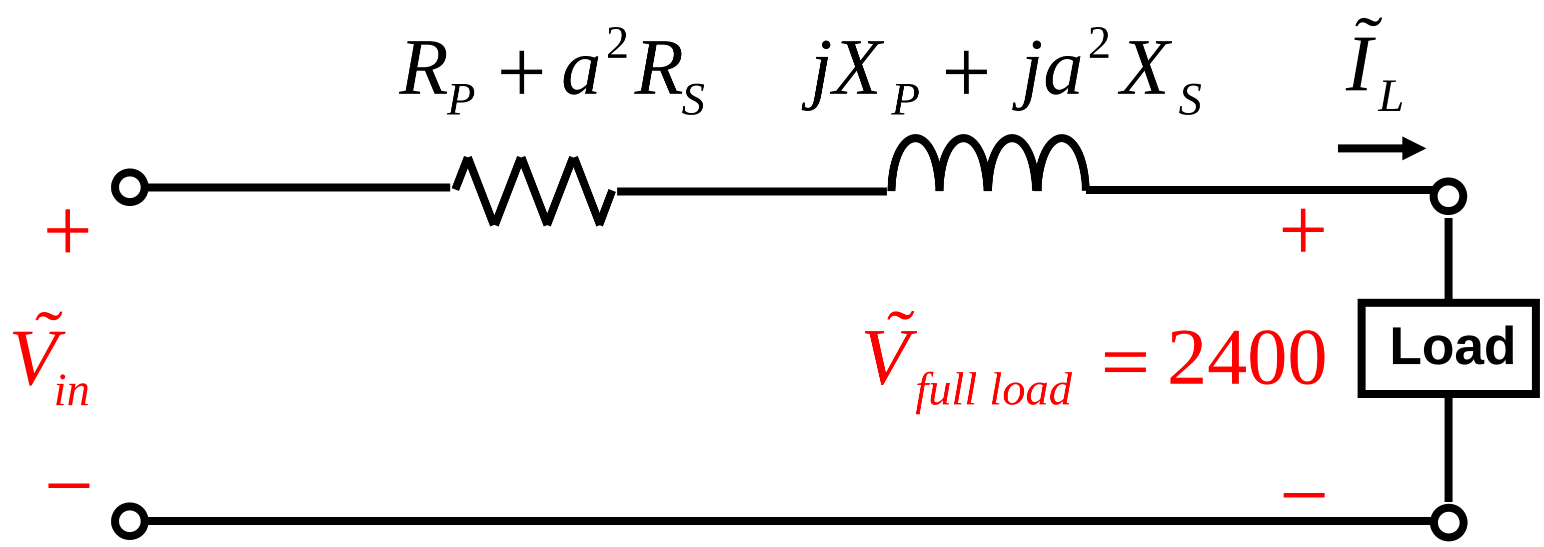
$$= 16.66 - j12.5$$

Example – Continued

Solution

The required input voltage is

$$\tilde{V}_{in} = Z_{eq}\tilde{I}_L + \tilde{V}_{full\ load}$$
$$= (1.42 + j1.82)(16.66 - j12.5) + 2400$$
$$= 2446 + j13\,V$$



Example – Continued **Solution**

The required input voltage is

$$\tilde{V}_{in} = Z_{eq}\tilde{I}_L + \tilde{V}_{full\ load}$$

$$= (1.42 + j1.82)(16.66 - j12.5) + 2400$$

$$= 2446 + j13\,V$$

$$\left|\tilde{V}_{in}\right| = 2446$$

Example – Continued **Solution**

Under loaded conditions, the load voltage is 2400 volts, by design. If the load were removed the load voltage would jump to 2446 volts. Thus

$$VR = \frac{V_{Secondary,\, no\ load} - V_{Secondary,\ full\ load}}{V_{Secondary,\ full\ load}} \times 100\%$$

$$= \frac{2446 - 2400}{2400} \times 100\%$$

$$= 1.92\%$$

The Per–Unit System – An Industry-Standard Normalization

Computations relating to machines and transformers are often carried out in *per-unit* form. This is where all quantities expressed as fractions of appropriately chosen *base values*.

All the usual computations are then carried out in these per unit values instead of the familiar volts, amperes, ohms, etc.

There are a number of advantages to the system. One is that the parameter values of machines and transformers typically fall in a reasonably narrow numerical range when expressed in a per-unit system based upon their rating. The correctness of their values is thus subject to a rapid approximate check.

The Per–Unit System

A second advantage is that when transformer equivalent-circuit parameters are converted to their per-unit values, the ideal transformer turns ratio becomes 1:1 and hence the ideal transformer can be eliminated.

This greatly simplifies analyses. For complicated systems involving many transformers of different turns ratios, this simplification is a significant one in that a possible cause of serious mistakes is removed.

The Per–Unit System Quantifies such as voltage V, current I, power P, reactive power Q, voltamperes VA, resistance R, reactance X, impedance Z, conductance G, susceptance B, and admittance Y can be translated to and from per-unit form as follows:

$$Per-unit\ value = \frac{actual\ value}{base\ value\ of\ quantity}$$

Once V_{base} and I_{base} have been chosen, then

$$P_{base}, Q_{base}, VA_{base} = V_{base} I_{base}$$

$$R_{base}, X_{base}, Z_{base} = \frac{V_{base}}{I_{base}}$$

Clearly only two independent base values can be chosen arbitrarily.

The Per–Unit System

Typically VA_{base} and V_{base} are chosen, then the remaining quantities are uniquely established as per the previous slide.

The value of VA_{base} must be the same over the entire system of analysis.

When a transformer is encountered, the values of V_{base} differ on each side and should be chosen to have the same ratio as the turns ratio of the transformer.

The Per–Unit System

Usually the rated or nominal voltages of the respective sides are chosen as base values.

If the base voltages of the primary and secondary are chosen to be in the ratio of the turns of the ideal transformer, the per-unit ideal transformer will have a unity turns ratio and hence can be eliminated.

If these rules are followed, the procedure for performing system analyses in per-unit can be summarized as follows:

The Per–Unit System We will illustrate via example, but the procedure is:

1. Select a *VA* base and a base voltage at some point in the system.

2. Convert all quantities to per unit on the chosen *VA* base and with a voltage base that transforms as the turns ratio of any transformer which is encountered as one moves through the system.

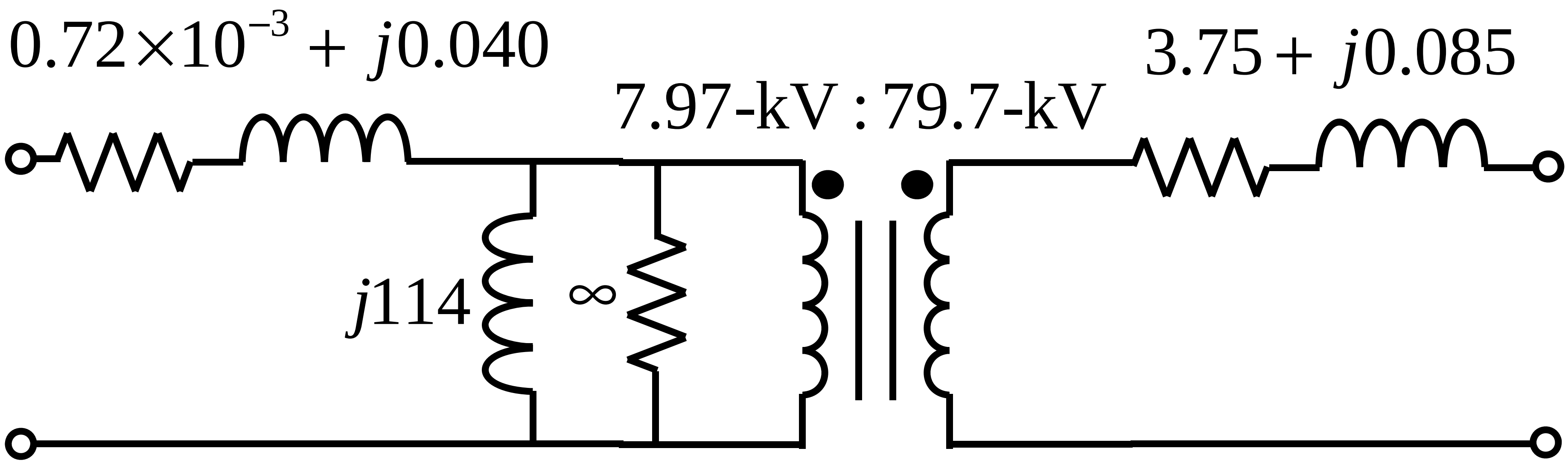
3. Perform a standard electrical analysis with all quantities in per unit.

4. When the analysis is completed, all quantities can be converted back to real units (e.g., volts, amperes, watts, etc.) by multiplying their per-unit values by their corresponding base values.

5. The procedure is best illustrated with an example.

The Per–Unit System – Example

The equivalent circuit for a 100-MVA, 7.97-kV:79.7-kV transformer is shown below. Convert this equivalent circuit to per unit using the transformer rating as base.



The Per–Unit System – Example **Solution**

The base quantities for the transformer are:

Low-voltage side: VA_{base} = 100-MVA, V_{base} = 7.97-kV

$$R_{base} = X_{base} = \frac{V_{base}^{\;2}}{VA_{base}} = 0.635\ \Omega$$

High-Voltage side: VA_{base} = 100-MVA, V_{base} = 79.7-kV

$$R_{base} = X_{base} = \frac{V_{base}^{\;2}}{VA_{base}} = 63.5\ \Omega$$

The Per–Unit System – Example – Compute the per unit values:

Solution

$$X_P = \frac{0.04}{0.635} = 0.063$$

$$X_S = \frac{3.75}{63.5} = 0.0591$$

$$X_M = \frac{114}{0.635} = 180$$

$$R_P = \frac{7.6\times10^{-4}}{0.635} = 0.0012$$

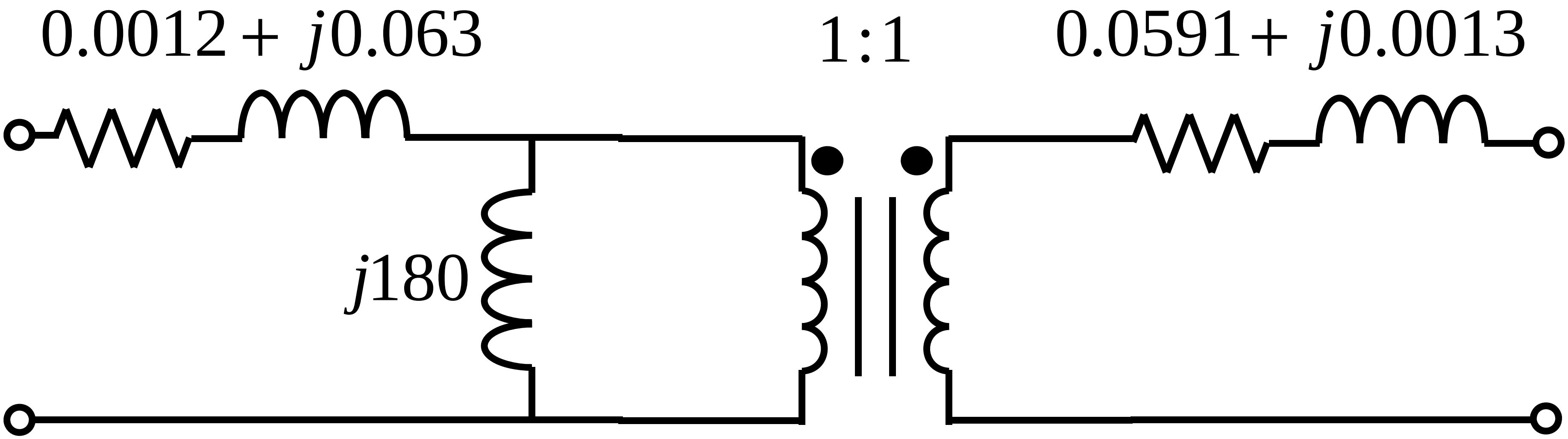
$$R_S = \frac{0.085}{63.5} = .0013$$

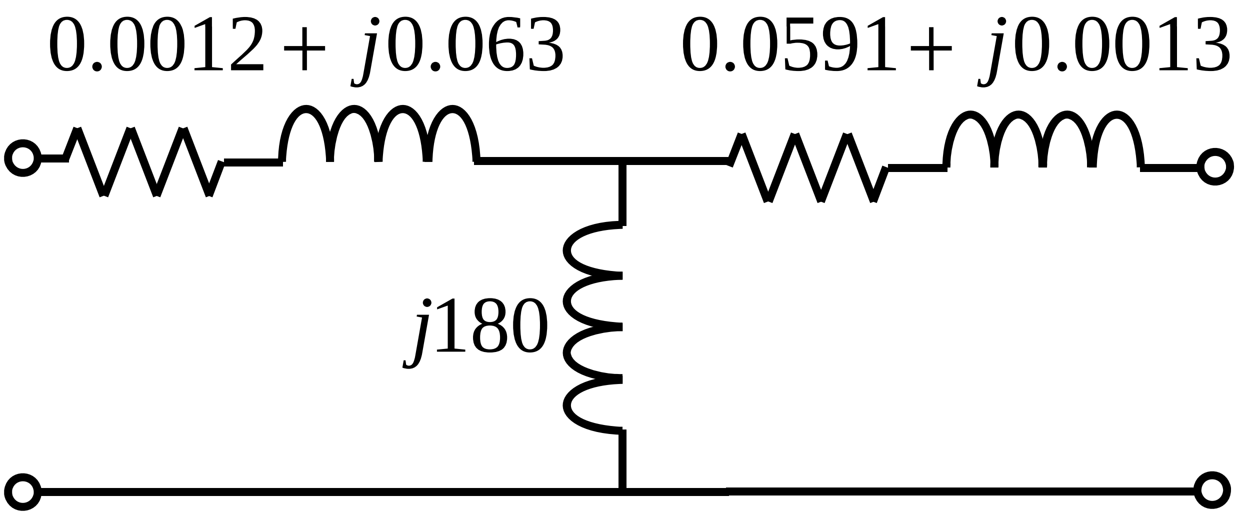
Compute the per unit turns ratio:

$$Turns\ Ratio_{pu} = \frac{7.97\text{-kV}}{7.97\text{-kV}} : \frac{79.7\text{-kV}}{79.7\text{-kV}} = 1:1$$

There is no need to keep the transformer as far as analysis is concerned.

The Per–Unit System – Example – *in per unit values* Solution





The Per–Unit System – Example

Consider again our earlier example of a 50-kVA, 2400:240-V transformer. We found earlier that the current on the low voltage side is 5.43 amps (Slide 96) and that its equivalent impedance referred to the high voltage side is 1.42 + j1.82 Ohms (Slide 98). Using the transformer rating as the base, express in per unit on the low- and high-voltage sides (a) the exciting current and (b) the equivalent impedance.

The base values are

$$V_{base,P} = 2400 \text{ V} \quad V_{base,S} = 240 \text{ V}$$

$$I_{base,P} = 20.8 \text{ A} \quad I_{base,S} = 208 \text{ A}$$

these give

$$Z_{base,P} = \frac{2400}{20.8} = 115.2\ \Omega \quad V_{base,S} = \frac{240}{208} = 1.152\ \Omega$$

The Per–Unit System – Example Solution

Using the values found in Slides 95 and 96,

$$I_{M,P} = \frac{0.543}{20.8} = 0.026 \ per\ unit$$

$$I_{M,S} = \frac{5.43}{208} = 0.026 \ per\ unit$$

As expected, the per-unit values are the same referred to either side corresponding to a unity turns ratio for the ideal transformer. This is a direct consequence of the choice of base voltages in the ratio of the transformer turns ratio and the choice of a constant volt-ampere base.

From Slide 100:

$$Z_{eq,P} = \frac{1.42 + j1.82}{115.2} = 0.0123 + j0.0158 \ per\ unit$$

$$Z_{eq,S} = \frac{0.0142 + j0.0182}{1.152} = 0.0123 + j0.0158 \ per\ unit$$

Again the values are the same, with the effect of the turns ratio accounted for by choice of base values.

The Per–Unit System – Example

A 15-kVA 120:460-V transformer has an equivalent series impedance of 0.018 + j0.042 per unit. Calculate the equivalent series impedance in ohms (a) referred to the low-voltage side and (b) referred to the high-voltage side.

Solution

$$Z_{base,L} = \frac{120}{15,000/120} = 0.96\ \Omega$$

$$Z_{base,H} = \frac{460}{15,000/460} = 14.1\ \Omega$$

The Per–Unit System – Example **Solution**

$$Z_{eq,L} = 0.96\left(0.018 + j0.042\right)\,\Omega$$
$$= 0.0173 + j0.0403$$
$$Z_{eq,H} = 14.1\left(0.018 + j0.042\right)$$
$$= 0.254 + j0.592\,\Omega$$

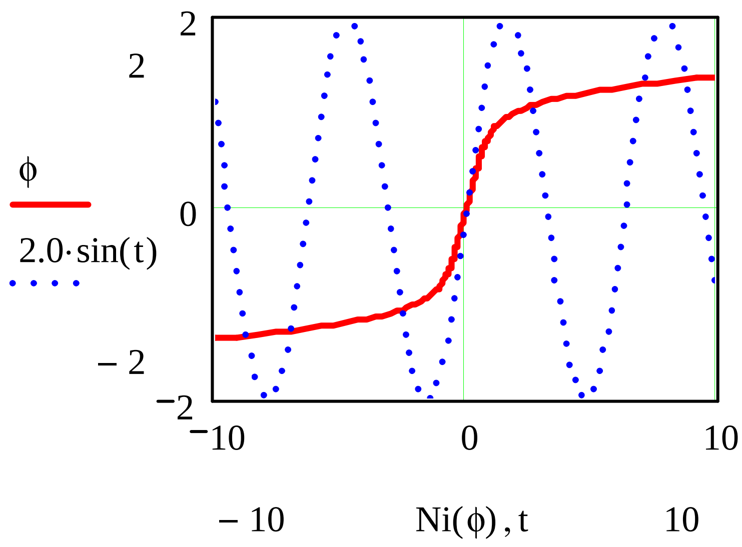
Transformer Ratings

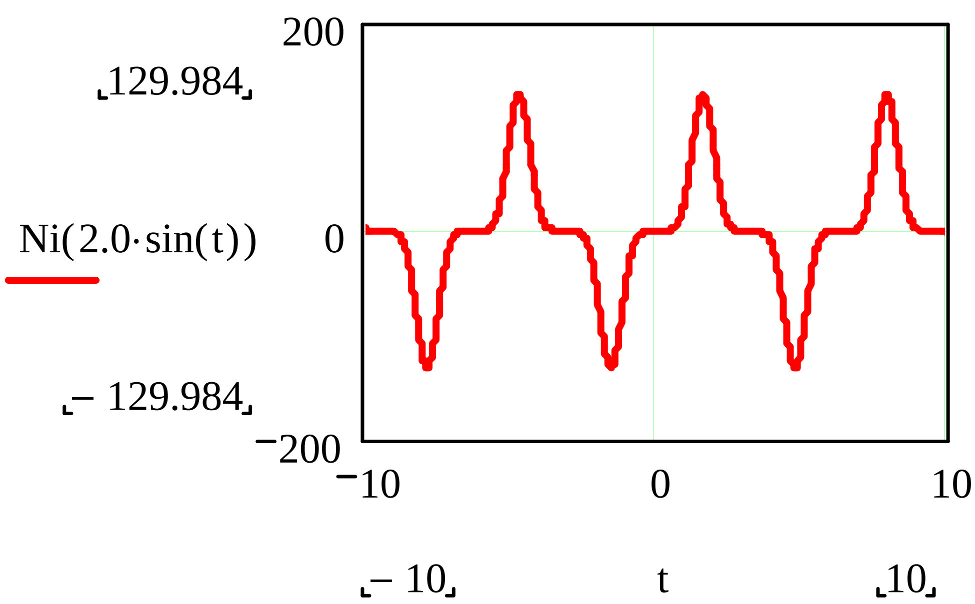
Transformers have four major ratings:

apparent power
voltage
current
frequency

Transformer Ratings

The voltage rating serves two purposes; the first is to insure that breakdown does not occur between the transformer windings. The second is prevent the core from becoming saturated which, as we have seen (see results of our earlier example below), can result in very large magnetization currents.





Transformer Ratings

If a steady-state voltage *v*(*t*) is applied to the transformer's primary winding,

$$v(t) = V_{\max} \sin \omega t$$

this gives the flux as

$$v(t) = N_P \frac{d\phi(t)}{dt} \Rightarrow \phi(t) = \frac{1}{N_P} \int v(t)\, dt$$

$$\phi(t) = -\frac{V_{\max}}{\omega N_P} \cos \omega t$$

Transformer Ratings

From

$$\phi(t) = -\frac{V_{\max}}{\omega N_P}\cos\omega t$$

we see that there are two ways to drive the core into saturation. One is to increase the voltage (our example). Another is to reduce the operating frequency.

Accordingly, a (European) transformer rated at 50 Hz may be operated at a 20% high voltage at 60 Hz assuming that this does not cause insulation breakdown.

Transformer Ratings

The purpose of the ***apparent power rating*** it that, together with the voltage rating, it sets the current flow through the windings. The current flow is important because it controls the i^2R losses which in turn controls the heating of the coils. Heating significantly reduces insulation lifetime.

The Problem of Current Inrush – a transient effect

Suppose that the moment the transformer is connected to the power line the voltage across the primary winding is,

$$v(t) = V_{\max} \sin(\omega t + \theta)$$

The maximum flux value reached during the first half-cycle of the applied voltage depends on the value of θ. For $\theta = 90^{o}$,

$$v(t) = V_{\max} \sin(\omega t + 90^\circ) = V_{\max} \cos \omega t$$

and

$$\phi(t) = \frac{V_{\max}}{\omega N_P} \sin \omega t \Rightarrow \phi_{max} = \frac{V_{\max}}{\omega N_P}$$

or just the maximum value at steady state.

The Problem of Current Inrush – a transient effect

Now suppose that $\theta = 0^{o}$, then $v(t) = V_{\max} \sin \omega t$

and the maximum flux during the first half-cycle is

$$\phi(t) = \frac{1}{N_P} \int_0^{\frac{T}{2}=\frac{1}{2f}=\frac{\pi}{\omega}} V_{\max} \sin \omega t dt$$

$$\phi(t) = -\frac{V_{\max}}{\omega N_P} \cos \omega t \Big|_0^{\frac{\pi}{\omega}}$$

$$\Rightarrow \phi_{\max} = \frac{2V_{\max}}{\omega N_P}$$

The Problem of Current Inrush – a transient effect

Now the maximum flux is twice its steady-state value. Recall again our earlier example showing what happens if, near saturation, the flux is further increased by even a small amount.

A doubling of flux could result (as a transient effect) is a ***huge*** inrush of current when the transformer is first plugged in. In fact, for this fist part of a cycle the primary really looks like a short.

The primary winding must be designed to handle this effect.

There is lots of engineering involved in designing magnetic circuits! This holds even truer for machines – our next topic.

Autotransformers (Text Section 3.9) and Three Phase Transformers (Text Section 3.10) are left as a reading exercise.